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Sur le mouvement plan du liquide visqueux compressible

par

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Présenté par H. STEINHAUS le 27 Janvier 1955

1. Soit S une courbe plane, simple, fermée qui sans se déformer se déplace parallèlement à une droite avec une vitesse constante, égale à U , dans un liquide visqueux, compressible, remplissant tout le plan à l'extérieur de S . Soit XY un système de coordonnées liées à la courbe S , l'axe X étant parallèle à la vitesse de S ; soient u, v — les composantes de la vitesse du liquide par rapport au système immobile $\bar{X}\bar{Y}$, parallèle au système XY ; soient p, ϱ, μ, ν — la pression, la densité et les coefficients de viscosité du liquide; u, v, p et ϱ sont les fonctions des coordonnées x, y dans le système XY . J'admetts que les forces extérieures n'existent pas.

J'admetts que: 1^o le liquide adhère à la courbe S ; on aura donc sur cette courbe $u=U, v=0$; 2^o l'énergie cinétique du mouvement est finie; 3^o il existe une suite de nombres positifs infiniment croissante a_n telle que

$$(1) \quad \int_0^{2\pi} d\varphi \int_{a_n}^{8a_n} \varrho r dr \leq M \cdot a_n^2,$$

$$(1^*) \quad \int_0^{2\pi} d\varphi \int_{a_n}^{8a_n} \frac{1}{\varrho} r dr \leq M \cdot a_n^2,$$

r et φ désignant les coordonnées polaires par rapport au système XY , M étant une constante *); 4^o u, v, p, ϱ et les dérivées de ces fonctions, qui apparaissent dans les équations du mouvement et dans l'équation de la continuité, sont continues **). Je démontre qu'il existe deux suites infiniment croissantes r_n et r'_n telles que les composantes P_x et P_y de la force exercée sur la courbe S par le liquide s'expriment par les formules

) Il suffit de poser $(1+\varepsilon)a_n$ au lieu de $8a_n$ dans les formules (1) et (1), ε étant une constante positive arbitraire.

**) Les conditions concernant u, v, p, ϱ peuvent être affaiblies.

$$(2) \quad P_x = -\lim_{n \rightarrow \infty} \left[\iint_{H_{r_n}} \frac{\partial(\varrho u)}{\partial t} d\sigma + r_n^2 \int_0^{2\pi} \frac{\partial(\varrho u_\varphi)}{\partial t} \sin \varphi d\varphi \right],$$

$$(3) \quad P_y = -\lim_{n \rightarrow \infty} \left[\iint_{H_{r_n}} \frac{\partial(\varrho v)}{\partial t} d\sigma - r_n^2 \int_0^{2\pi} \frac{\partial(\varrho u_\varphi)}{\partial t} \cos \varphi d\varphi \right];$$

u_r et u_φ désignant les composantes radiale et transversale de la vitesse du liquide par rapport au système $\bar{X}\bar{Y}$ et H , — la région contenue entre la courbe S et le cercle K , à rayon r et dont le centre est l'origine du système XY . Ces dernières formules ne résultent que des équations du mouvement et de l'équation de la continuité sans tenir compte de l'équation caractéristique du liquide, bien que la détermination du mouvement du liquide soit impossible sans cette dernière équation. Ces formules sont valables pour une fonction arbitraire $p(x, y, t)$ qui n'est pas liée avec la densité ϱ . Elles sont donc valables pour une vaste classe de mouvements, p. ex. pour le mouvement d'un liquide hétérogène, incompressible ou compressible et même dans le cas où l'équation caractéristique du liquide dépend du temps.

Des formules (2) et (3) il résulte que *le liquide n'exerce aucune force sur la courbe S dans le cas d'un mouvement permanent par rapport au système XY* .

2. Dès ce moment je me borne au cas où *la relation entre la densité et la pression est adiabatique* (ou au cas du *liquide incompressible*), c'est-à-dire que: I⁰ $\varrho = \text{const} \cdot p^\lambda$, $0 < \lambda < 1$. J'admetts de plus que: II⁰ *les conditions 1⁰-4⁰ sont satisfaites*, à l'exception de la condition (1), que je remplace par la condition plus restreinte: III⁰ *p est bornée*; IV⁰ u_r et $\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ sont bornées. Je démontre qu'il existe une suite infiniment croissante r_n'' telle que

$$(4) \quad U \cdot P_x = -\lim_{n \rightarrow \infty} \left[\frac{\partial r_n''}{\partial t} + D_{r_n''} + k \iint_{H_{r_n''}} \frac{p}{\varrho} \frac{\partial \varrho}{\partial t} d\sigma - U(k-1) r_n''^2 \int_0^{2\pi} \frac{\partial(\varrho u_\varphi)}{\partial t} \sin \varphi d\varphi \right],$$

où

$$(5) \quad D_r = \iint_{H_r} \left\{ \mu \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right\} d\sigma, \quad E_r = \frac{1}{2} \iint_{H_r} \varrho(u^2 + v^2) d\sigma, \quad k = \frac{1}{1-\lambda} \geq 1.$$

k — une constante > 1 .

Il résulte des conditions I⁰-IV⁰ et $\mu \neq 0$, que *le mouvement permanent du liquide par rapport au système XY n'existe pas*. En effet, dans ce cas $P_x = 0$ et, en tenant compte de (4), (5), toutes les dérivées partielles de u

et v disparaissent, ce qui donne d'après la condition 1^o: $u = U$, $v = 0$. Il résulte des équations du mouvement que p ainsi que ϱ sont constantes, ce qui est contraire à la condition 2^o (l'énergie cinétique est finie).

Si l'on envisage *le mouvement d'un liquide visqueux, incompressible, hétérogène* et si les conditions II^o-IV^o sont remplies, une légère modification de la démonstration nous conduit à la formule analogue à (4):

$$U \cdot P_x = - \lim_{n \rightarrow \infty} \left[\frac{\partial E_{r_n}'''}{\partial t} + D_{r_n}''' \right].$$

Il en résulte évidemment que le mouvement permanent par rapport au système XY n'existe non plus dans ce cas.

Les résultats de cette note sont une généralisation des résultats de W. Wolibner [1].

Le présent texte est un résumé du travail qui paraîtra dans *Studia Mathematica*.

INSTITUT MATHÉMATIQUE DE L'ACADEMIE POLONAISE DES SCIENCES

OUVRAGES CITÉS

[1] W. Wolibner, *Sur le mouvement plan du liquide visqueux, incompressible entourant une courbe simple fermée*, *Studia Mathematica* **12** (1951).

A Generalization of Newton's Method

by

M. ALTMAN

Presented by S. MAZUR on February 9, 1955

L. V. Kantorovitch [1] has shown that Newton's well-known classical iteration method can also be used for non-linear functional equations in Banach spaces. This method is defined by the formula

$$x_{n+1} = x_n - [P'(x_n)]^{-1} P(x_n),$$

where $P'(x)$ denotes the Fréchet differential of the non-linear continuous transformation $P(x)$, which appears in the equation $P(x)=0$. Several applications of this method are also given by him. Kantorovitch also investigated the following modification of Newton's method defined by the formula

$$x_{n+1} = x_0 - [P'(x_0)]^{-1} P(x_n),$$

where x_0 denotes the initial approach of the solution. Other authors have also examined Newton's method and its applications. However, in all these theorems it is supposed that the Fréchet differential possesses an inverse *). It appears that it is possible to define a process under a weaker condition such that the Fréchet differential is a linear transformation which maps a Banach space onto such a space. In particular, if the Fréchet differential possesses an inverse then the generalized process coincides with the above modification of Newton's method.

Let \mathfrak{X} and \mathfrak{Y} be two Banach spaces and let A be a linear (i. e. additive and continuous) transformation which maps \mathfrak{X} onto \mathfrak{Y} . Denote by θ the set of all solutions of the equation $A(x)=0$. We divide the space \mathfrak{X} into classes and we say that x_1 and x_2 belong to the same class \mathfrak{x} , if $x_1 - x_2 \in \theta$. This quotient space \mathfrak{X}/θ is a Banach space with the norm $\|\mathfrak{x}\| = \inf \|x\|$, $x \in \mathfrak{x}$. The transformation A defines in the space \mathfrak{X}/θ a linear transformation $y = \mathfrak{A}\mathfrak{x}$, where $x \in \mathfrak{x}$ and $y = Ax$. By a theorem of Banach the inverse transformation \mathfrak{A}^{-1} is also linear. The following lemma will be needed.

*) i. e. the linear transformation is a one-to-one mapping onto the whole space.

LEMMA. $\|\mathfrak{A}^{-1}\| = \|\bar{A}^{-1}\|$ *, where \bar{A} is the adjoint of A .

Proof. Notice that if ψ is a linear functional defined on \mathfrak{X} , such that $\psi(x) = 0$ for all $x \in \theta$, then the functional Ψ defined on \mathfrak{X}/θ by the formula

$$\Psi(\mathfrak{x}) = \psi(x), \quad \text{where } x \in \mathfrak{x},$$

is linear and $\|\Psi\| = \|\psi\|$.

Put $\psi = \bar{A}(\varphi)$ and $\Psi = \bar{\mathfrak{A}}(\varphi)$.

It follows from the definition of the adjoint transformation that

$$\varphi(y) = \varphi(Ax) = \psi(x) \quad \text{and} \quad \varphi(y) = \varphi(\mathfrak{A}\mathfrak{x}) = \Psi(\mathfrak{x}),$$

hence $\varphi(x) = 0$ for all $x \in \theta$ and by the above remark we have $\|\varphi\| = \|\Psi\|$. Since

$$\|\bar{\mathfrak{A}}^{-1}\| = \sup_{\|\Psi\| \leq 1} \|\bar{\mathfrak{A}}^{-1}(\Psi)\| = \sup_{\|\varphi\| \leq 1} \|\varphi\|$$

and

$$\|\bar{A}^{-1}\| = \sup_{\|\psi\| \leq 1} \|A^{-1}(\psi)\| = \sup_{\|\varphi\| \leq 1} \|\varphi\|,$$

we infer that $\|\bar{\mathfrak{A}}^{-1}\| = \|\bar{A}^{-1}\|$. Consequently $\|\mathfrak{A}^{-1}\| = \|\bar{A}^{-1}\|$, since $\|\bar{\mathfrak{A}}^{-1}\| = \|\bar{\mathfrak{A}}^{-1}\|$ and $\|\bar{\mathfrak{A}}^{-1}\| = \|\mathfrak{A}^{-1}\|$.

Now let $y = P(x)$ be a continuous transformation of \mathfrak{X} into \mathfrak{Y} . We suppose that $P(x)$ is differentiable in the sense of Fréchet in some sphere $S(x_0, r)$ with centre x_0 and radius r and that the differential of Fréchet $P'(x_0) = A$ maps the space \mathfrak{X} onto \mathfrak{Y} . Using the same notation as in the lemma we define two sequences as follows:

Let ε be an arbitrary positive number. Let us set

$$\mathfrak{x}_1 = -\mathfrak{A}^{-1}P(x_0).$$

There exists in \mathfrak{x}_1 an element x_1 such that $\|x_1\| \leq \|\mathfrak{x}_1\| + \varepsilon$. Let us set

$$\mathfrak{x}_2 = \mathfrak{x}_1 - \mathfrak{A}^{-1}P(x_0 + x_1).$$

There exists in \mathfrak{x}_2 an element x_2 such that $\|x_2 - x_1\| \leq \|\mathfrak{x}_2 - \mathfrak{x}_1\| \cdot (1 + \varepsilon)$. If the elements $x_1 \in \mathfrak{x}_1, \dots, x_{n-1} \in \mathfrak{x}_{n-1}$ are already defined, then we put

$$(1) \quad \mathfrak{x}_n = \mathfrak{x}_{n-1} - \mathfrak{A}^{-1}P(x_0 + x_{n-1})$$

and choose x_n from \mathfrak{x}_n such that $\|x_n - x_{n-1}\| \leq \|\mathfrak{x}_n - \mathfrak{x}_{n-1}\| \cdot (1 + \varepsilon)$.

THEOREM 1. Let us assume that the following conditions are satisfied:

$$\|P(x_0)\| \leq D, \quad \|\bar{A}^{-1}\| \leq B, \quad \|P'(x) - P'(x_0)\| \leq C$$

for all $x \in S(x_0, r)$,

$$BC < 1 \quad \text{and} \quad BD < r(1 - BC).$$

*) In this case the domain of \bar{A}^{-1} is not in general the whole space.

Then the sequence $\{x_0 + x_n\}$ defined by formula (1) converges to a solution x^* of the equation

$$P(x) = 0$$

and $x^* \in S(x_0, r)$.

Proof. Under the assumptions of theorem 1 there exists an arbitrarily small $\varepsilon > 0$ such that the conditions

$$(3) \quad BC(1 + \varepsilon) < 1 \quad \text{and} \quad BD + \varepsilon < r[1 - BC(1 + \varepsilon)]$$

are still satisfied. Now consider the sequence $\{x_n\}$ defined by formula (1), where the number ε satisfies conditions (3). Since $x_{n-1} \in \mathfrak{x}_{n-1}$, we have by the definition of the transformation \mathfrak{A}

$$\mathfrak{A}(\mathfrak{x}_{n-1}) = A(x_{n-1}) = P'(x_0)x_{n-1},$$

hence

$$\mathfrak{x}_n = -\mathfrak{A}^{-1}[P(x_0 + x_{n-1}) - P'(x_0)x_{n-1}]$$

and

$$\mathfrak{x}_{n-1} = -\mathfrak{A}^{-1}[P(x_0 + x_{n-2}) - P'(x_0)x_{n-2}].$$

Thus we obtain

$$\mathfrak{x}_n - \mathfrak{x}_{n-1} = -\mathfrak{A}^{-1}[P(x_0 + x_{n-1}) - P(x_0 + x_{n-2}) - P'(x_0)(x_{n-1} - x_{n-2})].$$

By using the lemma and the analogue of Lagrange's formula we get

$$\|\mathfrak{x}_n - \mathfrak{x}_{n-1}\| \leq B \sup \|P'(x_0 + \bar{x}) - P'(x_0)\| \cdot \|x_{n-1} - x_{n-2}\|,$$

where $\bar{x} = x_{n-2} + t(x_{n-1} - x_{n-2})$ and $0 \leq t \leq 1$. Consequently,

$$\|x_n - x_{n-1}\| \leq BC(1 + \varepsilon) \|x_{n-1} - x_{n-2}\|, \quad \text{for } n = 2, 3, \dots$$

It follows from conditions (3) that the sequence $\{x_0 + x_n\}$ is contained in the sphere $S(x_0, r)$ and converges to some element x^* which also belongs to $S(x_0, r)$. The sequence $\{\mathfrak{x}_n\}$ is convergent, since the sequence of $x_n \in \mathfrak{x}_n$ is equally so. Thus $P(x_0 + x_n) \rightarrow 0$, as $n \rightarrow \infty$ and x^* is a solution of equation (1).

Remark 1. If the set of all solutions x of the equation $P'(x_0)x = 0$ is a finite dimensional space, then in formula (1) the number ε may be replaced by zero. This follows from the fact that the minimum distance of a point from a finite dimensional Banach space is reached at some point of this space. In this case the condition

$$BD < r(1 - BC)$$

may be replaced by

$$BD \leq r(1 - BC).$$

Remark 2. If the Fréchet differential $P'(x_0)$ is a linear transformation of \mathfrak{X} onto \mathfrak{Y} having an inverse, then the process (1) coincides, evidently, with the so-called modified Newton's process of L. V. Kantorovich [1].

Let us assume that the Fréchet differential $P'(x)$ satisfies the Lipschitz condition in some sphere $S(x_0, r)$. Using the same method as in theorem 1 we can prove the following

THEOREM 2. *Let us suppose that the following conditions are satisfied: There exists a constant K such that*

$$1^o \quad B^2KD < \frac{1}{4}.$$

2^o The Fréchet differential $P'(x)$ satisfies the Lipschitz condition with the constant K in the sphere $S(x_0, r)$, where

$$r > \frac{1 - \sqrt{1 - 4B^2KD}}{2BK}$$

Then process (1) converges to a solution of equation (2). This solution belongs to sphere $S(x_0, r)$.

Remark 3. It is sufficient to assume that the Fréchet differential $P'(x)$ satisfies the Lipschitz condition in the sphere $S(x_0, r)$, where $r > 2BD$.

Let us now consider equation (2) under the assumption that there exists the second Fréchet differential $P''(x)$, the norm of which is bounded in some sphere $S(x_0, r)$, i. e. $\|P''(x)\| \leq K$ for $x \in S(x_0, r)$.

In this case we need to introduce some modification in the definition of the element x_1 of sequence (1) as follows:

If $x_0 \in x_0$, then we set

$$x_1 = x_0 - \mathfrak{A}^{-1}P(x_0)$$

and choose an element $x_1 \in x_1$ such that

$$\|x_1 - x_0\| \leq \|x_1 - x_0\| \cdot (1 + \varepsilon).$$

Everything else remains unchanged. Notice that this definition may also be introduced in the previous cases.

THEOREM 3. *Let us assume that the following conditions are satisfied: There exists a constant K such that*

$$1^o \quad B^2KD < \sqrt{2} - 1.$$

2^o The second Fréchet differential $P''(x)$ exists and satisfies the condition

$$\|P''(x)\| \leq K$$

in the sphere $S(x_0, r)$, where

$$r > \frac{1 + B^2KD - \sqrt{1 - B^2KD - B^4K^2D^2}}{2BK}.$$

Then process (1) converges to a solution of equation (2). This solution belongs to the sphere $S(x_0, r)$.

Remark 4. It is sufficient to assume that the second Fréchet differential $P''(x)$ satisfies condition 2^o for $r > 2BD$.

Remark 5. If the set of all solutions x of the equation $P'(x_0)x = 0$ forms a finite dimensional space, then the sign < 0 (resp. > 0) in conditions 1^o and 2^o of theorems 2 and 3 and in remark 3 may be replaced by the sign < 0 (resp. > 0).

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Problème aux limites de Poincaré généralisé

par

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Présenté par S. MAZUR le 8 Février 1955

Le problème aux limites de Poincaré [1] consiste à trouver une fonction $u(x, y)$ harmonique à l'intérieur d'une courbe fermée L et satisfaisant en tout point de cette courbe à la relation linéaire

$$(1) \quad \frac{du}{dn} + a(s)u + b(s)\frac{du}{ds} = f(s)$$

entre les valeurs limites de la dérivée suivant la normale du/dn , celles de la dérivée tangentielle du/ds et celles de la fonction u elle-même, $a(s)$, $b(s)$ et $f(s)$ étant des fonctions données de la longueur d'arc s de la courbe donnée L qui y détermine la position du point mobile. Poincaré a résolu ce problème dans le cas particulier $a=0$, les fonctions données $b(s)$ et $f(s)$, de même que la courbe L étant supposées analytiques. L'auteur de cette communication a résolu le problème dans le cas $a\neq 0$, mais sous la même hypothèse d'analyticité [2]. Le mathématicien soviétique Hvedélidzé [3] l'a résolu complètement sous l'hypothèse plus générale, à savoir que les fonctions $a(s)$, $b(s)$ et $f(s)$ satisfont à la condition d'Hölder.

Cette communication contient la solution du problème de trouver une fonction $u(x, y)$ harmonique à l'intérieur d'un domaine D et qui, en tout point (s) de la frontière L de ce domaine, satisfait à la relation généralisée suivante:

$$(2) \quad \frac{du}{dn} + a(s)u = \lambda F\left(s, u, \frac{du}{ds}\right),$$

où la fonction $F(s, u, v)$ est définie dans une région fermée de trois variables réelles et λ est un paramètre.

Les hypothèses admises sont les suivantes:

I. La courbe fermée de Jordan, L , a une tangente continue en tout point; l'angle de cette tangente avec une direction fixe satisfait à la condition d'Hölder, c'est-à-dire

$$(3) \quad |\delta_{ss_1}| < \text{const } |s - s_1|^\gamma, \quad \text{où } 0 < \gamma < 1,$$

δ_{ss_1} désignant l'angle entre tangentes en deux points de la courbe L aux coordonnées curvilignes s et s_1 .

II. La fonction $F(s, u, v)$ de trois variables réelles, définie dans la région fermée

$$(4) \quad s \in L; \quad |u| \leq R_1; \quad |v| \leq R_2,$$

satisfait à la condition d'Hölder par rapport aux variables s, u et à la condition de Lipschitz par rapport à la variable v :

$$(5) \quad |F(s, u, v) - F(s_1, u_1, v_1)| \leq \text{const} [|s - s_1|^\alpha + |u - u_1|^\beta + |v - v_1|].$$

III. La fonction $a(s)$, définie sur L , est assujettie à la condition d'Hölder

$$(6) \quad |a(s) - a(s_1)| \leq \text{const} |s - s_1|^\alpha.$$

Admettons enfin que les exposants d'Hölder satisfont aux inégalités

$$(7) \quad \alpha < \beta \leq 1 \quad \text{et} \quad \alpha < \gamma \leq 1.$$

La fonction u est cherchée sous la forme d'un potentiel logarithmique de la couche simple de densité $\mu(\sigma)$:

$$(8) \quad u(M) = \int_L \log \frac{1}{r_{M\sigma}} \mu(\sigma) d\sigma.$$

On a alors

$$(9) \quad \left(\frac{du}{dn} \right)_s = -\pi \mu(s) + \int_L \frac{\sin \varphi_{s\sigma}}{r_{s\sigma}} \mu(\sigma) d\sigma \quad \text{et} \quad \frac{du}{ds} = \int_L \frac{\cos \varphi_{s\sigma}}{r_{s\sigma}} \mu(\sigma) d\sigma,$$

où $\varphi_{s\sigma}$ désigne l'angle entre le vecteur $r_{s\sigma}$ joignant le point (s) au point (σ) de L avec la direction positive de la tangente à L au point (s) . La fonction sous la première des intégrales (9) a une singularité faible et celle sous la seconde — une singularité forte si $r_{s\sigma} \rightarrow 0$; par conséquent, la seconde intégrale a une valeur principale de Cauchy. En substituant les membres droits de (9) dans la relation limite (2), on arrive à l'équation suivante pour la fonction inconnue $\mu(s)$:

$$(10) \quad \begin{aligned} -\pi \mu(s) + \int_L \frac{\sin \varphi_{s\sigma}}{r_{s\sigma}} \mu(\sigma) d\sigma + a(s) \int_L \log \frac{1}{r_{s\sigma}} \cdot \mu(\sigma) d\sigma \\ = \lambda F \left[s, \int_L \log \frac{1}{r_{s\sigma}} \cdot \mu(\sigma) d\sigma, \int_L \frac{\cos \varphi_{s\sigma}}{r_{s\sigma}} \cdot \mu(\sigma) d\sigma \right]. \end{aligned}$$

C'est une équation intégrale non-linéaire à une singularité forte. La méthode classique des approximations successives est impuissante à résoudre l'équation (10). Elle sera résolue par l'application du théorème topologique suivant [4]:

“Si, dans un espace linéaire, complet et normé, une transformation continue transforme un ensemble fermé et convexe en son sous-ensemble compact, il existe au moins un point invariant de cette transformation”.

Nous nous appuyerons en outre, sans les démontrer ici, sur deux propriétés suivantes des intégrales figurant dans l'équation (10):

I. Si la fonction $\mu(\sigma)$, définie sur la courbe L , satisfait à la condition d'Hölder avec exposant α inférieur à l'unité, la fonction Φ , définie par l'intégrale ayant une *singularité forte*

$$(11) \quad \Phi(s) = \int_L \frac{\cos \varphi_{s\sigma}}{r_{s\sigma}} \mu(\sigma) d\sigma,$$

est définie sur L et satisfait aussi à la condition d'Hölder avec le même exposant α :

$$(12) \quad |\Phi(s) - \Phi(s_1)| \leq \text{const} |s - s_1|^\alpha.$$

II. Si la fonction $\mu(\sigma)$ est *continue* sur L , les intégrales à *singularité faible*

$$(13) \quad J_1(s) = \int_L \log \frac{1}{r_{s\sigma}} \cdot \mu(\sigma) d\sigma \quad \text{et} \quad J_2(s) = \int_L \frac{\sin \varphi_{s\sigma}}{r_{s\sigma}} \mu(\sigma) d\sigma$$

sont des fonctions définies sur L , satisfaisant à la condition d'Hölder, à savoir: la première avec un exposant arbitraire $h < 1$; la seconde avec un exposant arbitraire inférieur à l'exposant γ qui figure dans (3).

Pour résoudre l'équation (10), considérons l'espace fonctionnel E de toutes les fonctions *continues* $\mu(s)$, définies sur la courbe L , la distance entre deux points $\{\mu_1(s)\}$ et $\{\mu_2(s)\}$ de cet espace étant définie par la formule

$$(14) \quad \delta(\mu_1, \mu_2) = \sup |\mu_1(s) - \mu_2(s)|$$

et la somme de deux points, de même que le produit de point par nombre réel y étant aussi entendues au sens usuel. L'espace E est complet, normé et linéaire. Soit $S(\varrho, \varkappa)$ un ensemble fermé et borné des points $\{\mu(s)\}$ de l'espace E qui satisfont à l'inégalité

$$(15) \quad |\mu(s)| \leq \varrho$$

et à la condition d'Hölder avec l'exposant $\alpha < 1$ admis dans (5)

$$(16) \quad |\mu(s) - \mu(s_1)| \leq \varkappa |s - s_1|^\alpha.$$

Les constantes positives ϱ et \varkappa sont supposées fixées suffisamment petites pour que l'on ait

$$(17) \quad \left| \int_L \log \frac{1}{r_{s\sigma}} \mu(\sigma) d\sigma \right| \leq R_1 \quad \text{et} \quad \left| \int_L \frac{\cos \varphi_{s\sigma}}{r_{s\sigma}} \mu(\sigma) d\sigma \right| \leq R_2.$$

L'ensemble $S(\varrho, \varkappa)$ est convexe. En effet, si $\{\mu_1\}$ et $\{\mu_2\}$ sont deux points assujettis à (15) et (16), le point $\{(1-\varkappa)\mu_1 + \varkappa\mu_2\}$, où $0 < \varkappa < 1$, l'est aussi. Transformons l'ensemble $S(\varrho, \varkappa)$ en faisant correspondre à tout point $\{\mu(s)\}$ de cet ensemble un point $\psi(s)$ déterminé par l'équation

$$(18) \quad \begin{aligned} -\pi\psi(s) + \int_L \frac{\sin \varphi_{s\sigma}}{r_{s\sigma}} \psi(\sigma) d\sigma + a(s) \int_L \log \frac{1}{r_{s\sigma}} \psi(\sigma) d\sigma \\ = \lambda F \left[s, \int_L \log \frac{1}{r_{s\sigma}} \cdot \mu(\sigma) d\sigma, \int_L \frac{\cos \varphi_{s\sigma}}{r_{s\sigma}} \mu(\sigma) d\sigma \right]. \end{aligned}$$

En supposant que la fonction $a(s)$ ne soit pas positive, la fonction $\psi(s)$ vient correspondre à la fonction μ d'une façon univoque, d'après le premier théorème de Fredholm. En outre la fonction ψ est continue. Puis, d'après la propriété I. de l'intégrale (11) et d'après (5), le membre droit de l'équation (18) satisfait à la condition d'Hölder avec l'exposant α qui y est admis; de même, d'après la propriété II. des intégrales (13), les intégrales figurant dans le membre gauche de l'équation (18) satisfont aussi à la condition d'Hölder avec l'exposant α . Il en résulte que la fonction $\psi(s)$, qui correspond à la fonction $\mu(s)$, satisfait à la condition d'Hölder avec le même exposant α . Enfin, si la valeur absolue du paramètre λ est suffisamment petite, on a pour la fonction $\psi(s)$

$$(19) \quad |\psi(s)| < \varrho \quad \text{et} \quad |\psi(s) - \psi(s_1)| < \varkappa |s - s_1|^\alpha.$$

Le transformé S' des points $\{\psi(s)\}$ de S fait donc partie de S . On peut montrer que la transformation fonctionnelle (18) est continue et que le transformé S' est compact. En vertu du théorème précité de Schauder, il existe par conséquent un point invariant $\mu^*(s)$ de la transformation (18), donc une solution de l'équation intégrale (10). L'intégrale (8) fournit ensuite la solution cherchée du problème généralisé de Poincaré.

Un développement plus détaillé du problème paraîtra sous le même titre dans Annales Polonici Mathematici.

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Generalizations of the Theorems on Paradoxical Decompositions of the Sphere

by

JAN MYCIELSKI

Presented by W. SIERPIŃSKI on February 15, 1955

In paper [1] Professor Sierpiński has effectively proved the following theorem:

There exists a free group of rotations of the sphere of power 2^{\aleph_0} .

Making use of this theorem, I have generalized the results of R. M. Robinson [2] and of J. F. Adams [3] in which they used only an older theorem of Hausdorff, stating that there exists a free group of rotations of the sphere with two free generators. The generalizations of the results of paper [2] lead to the following theorems:

THEOREM 1. *There exists such a set E on the sphere ($x^2 + y^2 + z^2 = 1$) that for every cardinal number n satisfying the inequalities $2 < n \leq 2^{\aleph_0}$ this sphere may be decomposed into n disjoint parts each of which is congruent with E by rotation around the centre of the sphere.*

(From this theorem we can draw several obvious conclusions concerning paradoxical decompositions of the sphere).

THEOREM 2. *If $1 < n \leq 2^{\aleph_0}$, then the solid sphere S ($x^2 + y^2 + z^2 \leq 1$ or $x^2 + y^2 + z^2 < 1$) may be decomposed into n disjoint parts, one of which is $\frac{1}{n}S$ and all the others are $\frac{1}{n}S$ *).*

(This theorem is, at the same time, a strengthening of the main theorem of paper [1]).

The generalization of the theorem of paper [3] runs as follows:

THEOREM 3. *Let $0 < M \leq 2^{\aleph_0}$, $0 < \bar{N} \leq 2^{\aleph_0}$, and let $\{P_\mu\}_{\mu \in M}$, $\{Q_\mu\}_{\mu \in M}$ be families of non-empty proper subsets of set N .*

*) $\frac{1}{n}$ denotes congruence of sets by decomposition into n parts. For exact explanation see, for instance, [1] p. 235.

Then the sphere may be decomposed into N disjoint parts $\{A_v\}_{v \in N}$, satisfying the system of congruences

$$\sum_{v \in P_\mu} A_v \simeq \sum_{v \in Q_\mu} A_v \quad \mu \in M^*).$$

This theorem implies the following corollary:

COROLLARY. The sphere may be decomposed into a sequence of sets A_1, A_2, \dots , satisfying every congruence of the form

$$\sum_{n \in N_1} A_n \simeq \sum_{n \in N_2} A_n,$$

where N_1 and N_2 are arbitrary non-empty proper subsets of the set of natural numbers.

This note presents the results of the author's paper entitled *On the Paradox of the Sphere*, which will be published in *Fundamenta Mathematicae*.

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^{*}) \simeq denotes congruence of sets by rotation, or by rotation and reflection (reflection playing an important rôle in this theorem).

On the Infinite Sums of Models

by

J. ŁOŚ and R. SUSZKO

Presented by K. KURATOWSKI on February 18, 1955

We consider models $\mathfrak{M} = \langle A_1 R_1, R_2, \dots, F_1, F_2, \dots \rangle$ in some given elementary theory *). Let $\{\mathfrak{M}_n\}_{n=1,2,\dots}$ be an increasing (nondecreasing) sequence of models and let $\mathfrak{M} = \sum_{n=1}^{\infty} \mathfrak{M}_n$ **). A formula α is said to be persistent in the sequence $\{\mathfrak{M}_n\}_{n=1,2,\dots}$ if either α is valid in the model \mathfrak{M} or α is not valid in some model \mathfrak{M}_n .

We shall look for the conditions under which a formula α is persistent in every sequence of models.

We limit our consideration to the field of sentences, i. e. formulas without free variables. A sentence α is called a $\Pi\Sigma$ -sentence if there is a sentence β of the form

$$\prod_{x_{i_1}} \dots \prod_{x_{i_n}} \sum_{x_{j_1}} \dots \sum_{x_{j_m}} \gamma(x_{i_1}, \dots, x_{i_n}, x_{j_1}, \dots, x_{j_m}) \quad (n, m = 0, 1, 2, \dots),$$

containing only the indicated quantifiers and such that the equivalence $\alpha \equiv \beta$ is a tautology.

THEOREM 1. *A formula α is persistent in every sequence of models if, and only if, α is a $\Pi\Sigma$ -sentence.*

The proof is based on the theorems of [1] and the following lemma:

Lemma. *If B is a Boolean algebra, $b \in B$ and D is an additive and multiplicative subset of B such that $b \in D$, then there exist a prime ideal I_0 and an ideal I of B such that*

$$I \neq B, \quad b \in I_0, \quad b' \in I \quad \text{and} \quad D \cap I_0 \subset I.$$

*) For notions and notations see [1].

**) The sequence $\{\mathfrak{M}_n\}_{n=1,2,\dots}$ is called increasing if for every n , \mathfrak{M}_n is a submodel of \mathfrak{M}_{n+1} . The sum $\sum_{n=1}^{\infty} \mathfrak{M}_n$ is then the least model, of which every \mathfrak{M}_n is a submodel.

From theorem 1 follows:

THEOREM 2. *Let $\mathfrak{U}_0 = \mathfrak{U}(X)$ be a definable class of models. The following conditions are equivalent:*

a) *the sum of every increasing sequence of models belonging to \mathfrak{U}_0 also belongs to \mathfrak{U}_0 .*

b) *there exists a set Y of $\Pi\Sigma$ -sentences such that*

$$\mathfrak{U}_0 = \mathfrak{U}(Y) \text{ (consequently } Cn(X) = Cn(Y)).$$

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On the Maximum Modulus of Univalent Functions

by

J. KRZYŻ

Presented by K. KURATOWSKI on February 14, 1955

Introduction

Let S denote the class of functions regular and univalent in the unit circle $|z|<1$ and such that $f(z)=z+a_2z^2+\dots$ If $M(r,f)=\max|f(z)|$ ($|z|=r<1$), then the function $M(r,f)(1-r)^2r^{-1}$ decreases and therefore tends to a limit $\alpha<1$ as $r\rightarrow 1$ for any $f\in S$. The case of equality, $\alpha=1$, occurs only if $f(z)$ is a Koebe function: $f_0(z)=z(1-ze^{i\varphi})^{-2}$. This result is due to W. K. Hayman (see [1], p. 177 and [2], p. 1624).

M. Biernacki remarked that the fact that $M(r,f)(1-r)^2r^{-1}$ decreases is an immediate consequence of the bound for the logarithmic derivative of $f(z)$. The method due to Biernacki is contained in the lemma proved below. We next give an alternative proof of Hayman's result for univalent functions (due to Biernacki) and prove that the limit

$$\lim_{r\rightarrow 1} M(r,f')(1-r)^3=\beta \quad (f\in S)$$

exists and, besides, that $\beta=2\alpha$. This means that, if the latter limit is different from zero, then $M(r,f')$ asymptotically approaches $\frac{2M(r,f)}{1-r}$, as $r\rightarrow 1$.

In the sequel the term "decreasing function" means that $r_1< r_2$ implies $f(r_1)\geq f(r_2)$. If the equality is excluded the function will be called strictly decreasing.

A lemma on asymptotic growth of maximum modulus

LEMMA. *If $f(z)$ is regular for $|z|<R$ and on every circle $|z|=r<R$ there is*

$$\left| \frac{f'(z)}{f(z)} \right| < \frac{d}{dr} \log \psi(r) = \frac{\psi'(r)}{\psi(r)},$$

where $\psi(r)$ is a real, positive and differentiable function of $r \in (0, R)$, then (θ being fixed) $\frac{|f(re^{i\theta})|}{\psi(r)}$ and also $\frac{M(r, f)}{\psi(r)}$ are decreasing functions of $r \in (0, R)$.

The limits $\lim_{r \rightarrow R} \frac{M(r, f)}{\psi(r)}$, $\lim_{r \rightarrow R} \frac{|f(re^{i\theta})|}{\psi(r)}$ (θ being fixed) exist.

Proof. We obviously have

$$\frac{d}{dr} \log |f(re^{i\theta})| = \frac{\frac{d}{dr} |f(re^{i\theta})|}{|f(re^{i\theta})|} < \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| < \frac{d}{dr} \log \psi(r).$$

Hence $\frac{d}{dr} \log \frac{|f(re^{i\theta})|}{\psi(r)} < 0$ and this means that $\frac{|f(re^{i\theta})|}{\psi(r)}$ decreases.

If $r_1 < r_2$ and $M(r_2, f) = |f(r_2 e^{i\theta})|$, then

$$\frac{M(r_2, f)}{\psi(r_2)} = \frac{|f(r_2 e^{i\theta})|}{\psi(r_2)} < \frac{|f(r_1 e^{i\theta})|}{\psi(r_1)} < \frac{M(r_1, f)}{\psi(r_1)},$$

the function $\frac{M(r, f)}{\psi(r)}$ decreases, too.

The existence of the limits mentioned is obvious.

If the case of equality is excluded, i. e. if $\left| \frac{f'(z)}{f(z)} \right| < \frac{\psi'(r)}{\psi(r)}$, then $\frac{|f(re^{i\theta})|}{\psi(r)}$ and $\frac{M(r, f)}{\psi(r)}$ are strictly decreasing.

COROLLARIES. The well-known inequality for the logarithmic derivative of $f \in S$ (see [4], p. 86):

$$\left| \frac{f'(z)}{f(z)} \right| < \frac{1+r}{r(1-r)} = \frac{d}{dr} \log \frac{(1-r)^2}{r}$$

(with the sign of equality for $f = f_0$ only) at once gives Hayman's result mentioned above.

If $f \in S$, then we have

$$\left| \frac{f''(z)}{f'(z)} \right| < \frac{2r+4}{1-r^2} = \frac{d}{dr} \log \frac{1+r}{(1-r)^3}$$

(with equality for $f = f_0$ only, see e. g. [3], p. 54).

This implies that $|f'(re^{i\theta})|(1-r)^3(1+r)^{-1}$ and also $M(r, f)(1-r)^3(1+r)^{-1}$ decrease (strictly decrease if $f \neq f_0$).

Since $\lim_{r \rightarrow 0} M(r, f')(1-r)^3(1+r)^{-1} = 1$, we have

$$\lim_{r \rightarrow 1} M(r, f')(1-r)^3(1+r)^{-1} = \frac{1}{2} \lim_{r \rightarrow 1} M(r, f')(1-r)^3 = \frac{1}{2} \beta < 1$$

with the equality $\beta = 2$ for the Koebe function $f = f_0$ only.

We shall now prove that for every $f \in S$ $2a = \beta$, where

$$a = \lim_{r \rightarrow 1} M(r, f)(1-r)^2, \quad \beta = \lim_{r \rightarrow 1} M(r, f')(1-r)^3$$

Proof that $2\alpha=\beta$.

We have $\left| \frac{f'(z)}{f(z)} \right| \leq \frac{1+r}{r(1-r)}$ ($|z|=r < 1$), and this implies $\frac{M(r, f')}{M(r, f)} \leq \frac{1+r}{r(1-r)}$

or

$$(3.1) \quad (1+r) M(r, f) - r(1-r) M(r, f') \geq 0.$$

Besides,

$$\alpha - \frac{1}{2} \beta = \lim_{r \rightarrow 1} (1-r)^2 \frac{(1+r) M(r, f) - r(1-r) M(r, f')}{r(1+r)} \geq 0$$

by (3.1). This means that

$$(3.2) \quad 2\alpha \geq \beta.$$

If $\alpha=0$, then $\beta=0$ by (3.2). For $\alpha > 0$ Hayman obtained (see [2], p. 1625) the following representation for $f \in S$. Let ε be an arbitrary positive constant not exceeding 1 and suppose that $|f(z)|$ attains its maximum value ($|z|$ being fixed) at $z=re^{i\theta}$ and that $M(r, f) e^{i\lambda} = f(re^{i\theta})$. Then for $r_0(\varepsilon) < r < 1$

$$(3.3) \quad f(z) = \frac{ae^{i\lambda}}{(1-ze^{-i\theta})^2} [1 + \varepsilon(z)],$$

where $|\varepsilon(z)| < \varepsilon$ for all $z \in D$, D being the “quadrilateral”:

$$\varepsilon(1-r) \leq 1 - |z| \leq \frac{1-r}{\varepsilon}, \quad |\theta - \arg z| \leq \frac{1-r}{\varepsilon}.$$

It is easy to see that for r sufficiently near 1 the circle K : $|z-re^{i\theta}| = (1-\varepsilon)(1-r)$ is contained in D . We have for $z=re^{i\theta}$ (the point at which $|f(z)| = M(r, f)$)

$$|\varepsilon'(z)| = \left| \frac{1}{2\pi i} \int_K \frac{\varepsilon(\zeta) d\zeta}{(\zeta-z)^2} \right| \leq \frac{\varepsilon 2\pi(1-\varepsilon)(1-r)}{2\pi(1-\varepsilon)^2(1-r)^2},$$

or

$$(1-r)|\varepsilon'(z)| \leq \frac{\varepsilon}{1-\varepsilon} < 2\varepsilon \quad \text{if } \varepsilon < 1/2.$$

Since ε is arbitrary, this implies that

$$(3.4) \quad \lim_{r \rightarrow 1} (1-r)|\varepsilon'(z)| = 0,$$

$z=re^{i\theta}$ being the point at which $|f(z)|$ attains its maximum. The function $\varepsilon(z)$ (depending on r) is defined by (3.3). For $z=re^{i\theta}$ we have from (3.3):

$$\frac{(1-r)f'(z)}{f(z)} = 2e^{i\theta} + \frac{(1-r)\varepsilon'(z)}{1+\varepsilon(z)}.$$

Hence, in view of (3.4) and $|\varepsilon(re^{i\theta})| < \varepsilon$:

$$(3.5) \quad \lim_{r \rightarrow 1} \frac{(1-r)|f'(re^{i\theta})|}{|f(re^{i\theta})|} = 2 \quad (\theta \text{ depends here on } r).$$

But

$$\frac{(1-r) M(r, f')}{M(r, f)} \geq \frac{(1-r) |f'(re^{i\theta})|}{|f(re^{i\theta})|}.$$

This and (3.5) imply:

$$2 \leq \lim_{r \rightarrow 1} \frac{(1-r)^3 M(r, f')}{(1-r)^2 M(r, f)} = \frac{\beta}{\alpha},$$

resp. $2\alpha \leq \beta$. Therefore, in view of (3.2), $2\alpha = \beta$ and this is the desired result. In particular, if $\alpha > 0$, then $\lim_{r \rightarrow 1} \frac{(1-r) M(r, f')}{M(r, f)} = 2$ for any $f \in S$.

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Lösbarkeit der Randwertaufgaben für allgemeine stark-elliptische Systeme mit Hilfe des Galerkin'schen Verfahrens

von

K. MAURIN

Vorgelegt von K. KURATOWSKI am 21. Februar 1955

In einer grundlegenden Abhandlung gelang es Lars Gårding [1] viele Randwertprobleme für allgemeine elliptische Gleichungen, ja sogar stark-elliptische Systeme, auf Gleichungen zweiter Art mit vollstetigen Operatoren in gewissen Hilbert'schen Räumen zurückzuführen und auf diese Weise allgemeine Existenzsätze auszusprechen.

Da für lineare Gleichungen mit vollstetigen Operatoren die abstrakte Galerkin'sche Methode [3], [5] oft anwendbar ist, lag es nahe in diesem konkreten Falle sie anzuwenden. Es erwies sich, dass die Kombination der abstrakten Methode mit der Gårding'schen Theorie zu den klassischen Galerkin'schen Gleichungen für die Ausgangsdifferentialoperatoren führt. Es gelang mir auf diese Weise zu zeigen, dass sogar im Falle des allgemeinen starkelliptischen Systems die Galerkin'sche Folge konvergiert und im Limes die *klassische* (also nicht nur eine verallgemeinerte) Lösung des Dirichlet'schen Problems liefert. Auch andere Randwertaufgaben lassen sich auf diese Weise behandeln.

Die in dieser Note gewonnenen Ergebnisse bilden eine weitgehende Verallgemeinerung und Verschärfung der Resultate von Michlin [3]. Es soll auf dieser Stelle bemerkt werden, dass der Gedanke, das Galerkin'sche Verfahren auch in diesem allgemeinen Falle anzuwenden, bei der Lektüre des vortrefflichen Referats von Michlin entstand. Man wird unschwer die Verwandschaft der beiden Methoden finden.

Definitionen

Wir bedienen uns der von L. Gårding benutzten Schreibweise.

Es sei Q ein allgemeiner linearer Differentialoperator $2m$ -ter Ordnung mit n -unabhängigen Veränderlichen:

$$Q = \sum_{|\beta| \leq 2m} q_\beta(x) D_\beta,$$

wobei $q_\beta(x)$ komplexwertigen Funktionen der n Veränderlichen im offenen beschränkten Bereich Ω_n des n -dimensionalen Raumes sind: $q_\beta \in C^{\beta+2m}(\Omega_n)$.

D_β Ableitungen $\left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n}$ der Ordnung $|\beta| = \beta_1 + \dots + \beta_n$.

Der Operator Q heisst elliptisch, wenn das Polynom

$$q_{2m}(x, \xi) = \sum_{|\beta|=2m} q_\beta(x) \xi_\beta \neq 0 \quad (\xi_\beta = \xi_1^{\beta_1} \cdots \xi_n^{\beta_n}),$$

wobei ξ_i reell, $\sum_{i=1}^n \xi_i^i > 0$. $D(Q)$ bedeutet Definitionsbereich von Q ; Q^* – der adjungierte (im Sinne der Theorie des Hilbert'schen Raumes) von Q .

Es werden zuerst elliptische Operatoren mit reelem Hauptteil

$$\sum_{|\alpha|=|\beta|=m} p_{\alpha\beta}(x) D_{\alpha+\beta},$$

untersucht wobei $p_{\alpha\beta}(x)$ reell und symmetrisch sind und $\inf_{|\xi|=1; |\alpha|=|\beta|=m} p_{\alpha\beta}(x) \xi_\alpha \xi_\beta$ eine positive untere Grenze in Ω_n besitzt.

$C_c^{2m}(\Omega_n)$ die Funktionen aus $C^{2m}(\Omega_n)$ mit kompakten Trägern.

$$(u, v)_j = \int_{\Omega_n} \sum_{|\beta|=j} D_\beta u(x) \overline{D_\beta v(x)} dx.$$

Durch Abschliessen von $C_c^{2m}(\Omega_n)$ in Bezug auf die Norm $|u| = (u, u)^{1/2}$ erhält man einen Hilbert'schen Raum \mathfrak{H} ; $\mathfrak{H}_0 = L^2(\Omega_n)$. Der Satz von Garding-Browder besagt, dass es ein $t_0 > 0$ gibt, dass für $t > t_0$, $(u, u)_{P_t}^{1/2} = \int_{\Omega_n} \left\{ \sum_{|\beta|=|\alpha|=m} p_{\alpha\beta}(x) D_\alpha u(x) \overline{D_\beta u(x)} + t u(x) \overline{u(x)} \right\} dx$ eine Norm des \mathfrak{H}_m bildet, die derjenigen $(|u|_m^2 + t|u|_0^2)^{1/2}$ äquivalent ist.

Das Dirichlet'sche Problem

Es sei Q ein allgemeiner elliptischer Operator mit demselben Hauptteil wie

$$P_t u(x) = (-1)^m \sum_{|\alpha|=|\beta|=m} D_\alpha (p_{\alpha\beta}(x) D_\beta u(x)) + t u(x),$$

d. h.: $Q = P_t + R_t$, wobei, $R_{-t} = \sum_{\substack{|\gamma| < 2m \\ r_\gamma(x) = \text{komplexwertig}}} r_\gamma(x) D_\gamma u(x) - t u(x)$, $t > t_0$

Es sei $g \in C^{2m}(\Omega_n)$ und $\left(\sum_{j=0}^m |g|_j^2 \right)^{1/2} < \infty$; $h \in L^2(\Omega_n)$ und dabei eine Hölderbedingung erfüllt.

Das (verallgemeinerte) Dirichlet'sche Problem besitzt folgende Form:

Gesucht eine solche Lösung u der Differentialgleichung

$$(2.1) \quad Qu = h,$$

dass

$$(2.2) \quad (g - u) \in \mathfrak{H}_m,$$

(d. h. u besitzt auf dem Rande von Ω_n dieselben Ableitungen der Ordnungen $0, \dots, m-1$, wie g). Wie Gårding gezeigt hat, ist das Dirichlet'sche Problem dem folgenden Problem gleichwertig:

Gesucht ist die Lösung g' der Gleichung

$$(2.3) \quad \tilde{g} = g' + \mathfrak{R}g',$$

wobei $g \in \mathfrak{H}_m$ und durch die Identität

$$(2.4) \quad (Qg, f)_0 - (h, f)_0 = (\tilde{g}, f)_{P_t} \quad \text{für alle } f \in C_c^{2m}(\Omega_n)$$

definiert wird.

\mathfrak{R} ist ein vollstetiger linearer Operator in \mathfrak{H}_m (und $\mathfrak{R}(\mathfrak{H}_m) \subset \mathfrak{H}_m$), definiert durch die Identität

$$(2.5) \quad (f, f')_{R_{-t}} = (\mathfrak{R}f, f')_{P_t}; \quad f, f' \in \mathfrak{H}_m;$$

$((f, f')_{R_0} = (R_0 f, f')$ wird durch entsprechende partielle Intergrationen zu einer bi-linearen Form $(f, f')_{R_0}$ auf $\mathfrak{H}_m \times \mathfrak{H}_m$ fortgesetzt).

Aus der Gleichung (2.3) bekommt Gårding eines der Hauptergebnisse seiner Theorie, das wir als den Satz von Gårding bezeichnen: wenn das homogene Problem (2.1), (2.2), bei $g, h = 0$, nur die triviale Lösung $u = 0$ besitzt, dann hat das unhomogene Problem: (2.1) $Qu = h$, (2.2) $(u - g) \in \mathfrak{H}_m$ die (einige) Lösung $u \in C^{2m}(\Omega_n)$ ((2.6) $u = g' - g$ nach der Korektur auf einer Nullmenge).

Der Satz von Gårding hat einen existentiellen Charakter. Wir zeigen im nächsten Punkte, dass die Lösung von (2.1), (2.2) mit Hilfe des Galerkin'schen Verfahrens gewonnen werden kann.

Lösbarkeit des verallgemeinerten Dirichlet'schen Problems mit Hilfe des Galerkinschen Verfahrens

Es sei $\varphi_1, \varphi_2, \dots$ ($\varphi_i \in D(Q)$, $i = 1, 2, \dots$) eine Basis von \mathfrak{H}_m ; H_k der auf $\varphi_1, \dots, \varphi_k$ aufgespannte Unterraum von \mathfrak{H}_m . E_k ist die orthogonale Projektion auf H_k , d. h. wenn Ψ_1, \dots, Ψ_k eine orthonormale Basis von H_k ist: $(\Psi_i, \Psi_l) = \delta_{il}$

$$(3.1) \quad E_k v = \sum_{i=1}^k (v, \Psi_i)_{P_t} \Psi_i; \quad v \in \mathfrak{H}_m.$$

Da \mathfrak{H}_m separabel ist, haben wir

$$(3.2) \quad \lim_{k \rightarrow \infty} |E_k v - v|_{P_t} = 0.$$

Jetzt formulieren wir unseren

SATZ 1. *Es seien die Voraussetzungen des Satzes von Gårding erfüllt (also das homogene Problem besitzt nur die Nulllösung), dann konvergiert die Folge*

$$(3.3) \quad u_k = g - g'_k,$$

wobei $g'_k = \sum_{i=1}^k a_i \varphi_i$, wobei die Koeffizienten a_i aus den Galerkin'schen Gleichungen

$$(3.4) \quad \sum_{i=1}^k a_i (Q\varphi_i, \varphi_v)_0 = (Qg - h, \varphi_v)_0 \quad v = 1, \dots, k$$

bestimmt (für grosse n auf die einzige Weise) werden.

Die Folge u_k konvergiert in der $\|\cdot\|_{P_t}$, also auch in der $\|\cdot\|_j$ Metrik ($j = 0, \dots, m$):

$$\lim_{n \rightarrow \infty} |u - u_k|_m = 0$$

u ist nach einer Korrektur auf der Nullmenge $2m$ -mal stetige Lösung des Dirichlet'schen Problems.

Beweis. Statt

$$(2.3) \quad g' + \Re g' - \tilde{g} = 0$$

zu lösen, lösen wir die k -dimensionale Approximationsgleichung:

$$(3.5) \quad E_k g'_k + E_k \Re g'_k - E_k \tilde{g} = 0; \quad (\text{da offenbar: } E_k g'_k = g'_k).$$

Da die Gleichung (2.3) eindeutig auflösbar ist,

$$\lim_{k \rightarrow \infty} |E_k \tilde{g} - \tilde{g}|_{P_t} = 0; \quad |E_k \Re - \Re|_{P_t} \xrightarrow{k \rightarrow \infty} 0, \quad \text{vgl. [5]}$$

besitzt nach einem klassischen Satz der Theorie der Fredholm'schen Gleichungen im Hilbert'schen Raum ($\Re, E_k \Re$ sind ja vollstetig) für genügend grosses k die Gleichung (3.5) eine einzige Lösung g'_k , wobei

$$(3.6) \quad \lim_{k \rightarrow \infty} |g' - g'_k|_{P_t} = 0,$$

wo g' die Lösung von (2.3) ist [5].

$(g - g' = u$ löst bekanntlich das Dirichlet'sche Problem). Nehmen wir eine orthonormale Basis Ψ_1, \dots, Ψ_k von H_k , dann ist (3.5) äquivalent den k -skalaren Gleichungen:

$$(3.7) \quad 0 = (E_k(g'_k + \Re g'_k - \tilde{g}), \Psi_v)_{P_t} = (g'_k + \Re g'_k - \tilde{g}, \Psi_v)_{P_t} \quad (v = 1, \dots, k),$$

da $E_k^* \Psi_v = E_k \Psi_v = \Psi_v$. Wegen (2.4) und (2.5), haben wir

$$(P_t g'_k, \Psi_v)_0 + (R_{-t} g'_k, \Psi_v)_0 = (Q g'_k, \Psi_v)_0 = (g, \Psi_v)_{P_t} = (Qg - h, \Psi_v)_0.$$

Wenn wir $g'_k = \sum_{i=1}^k b_i \Psi_i$ setzen, haben wir schliesslich:

$$(3.8) \quad (Q \sum_{i=1}^k b_i \Psi_i, \Psi_v)_0 = \sum_{i=1}^k b_i (Q \Psi_i, \Psi_v)_0 = (Qg - h, \Psi_v)_0.$$

Da der Ansatz $g''_k = \sum_{i=1}^k a_i \varphi_i$, wo a_1, \dots, a_k den Gleichungen

$$(3.4) \quad \sum_{i=1}^k a_i (Q \varphi_i, \varphi_v)_0 = (Qg - h, \varphi_v)_0, \quad v = 1, \dots, k$$

genügen, zu denselben $g'_k = g''_k$ führt [6], ist alles bewiesen, wenn wir $u_k = g - g'_k$ setzen, da nach (3.6)

$$|u - u_k|_{P_t} = |g - g' - g + g'_k|_{P_t} = |g' - g'_k|_{P_t} \rightarrow 0 \quad \text{für } k \rightarrow \infty.$$

Der Fall des starkelliptischen Systems. Andere Randwertaufgaben

Wie man sieht, war für unsere Beweisführung nur wesentlich, dass das Randwertproblem durch eine Fredholm'sche Gleichung, die eine einzige Lösung besass, ersetzt wurde. Wie aus den von Gårding ([1] 67, 68) gemachten Bemerkungen folgt, gilt der folgende

HAUPTSATZ: *Es sei $Qu = h$ ein starkelliptisches System mit regulären komplexwertigen Koeffizienten. Der Satz I gilt unverändert auch in diesem Falle. Die in der Fassung des Satzes, sowie die in der Beweisführung vorkommenden Vektoren bedeuten jetzt Funktionen, deren Werte dem r -dimensionalen unitären Raum angehören.*

Bemerkung: Starke Elliptizität bedeutet folgendes: Die $(r \times r)$ -Operatorenmatrix $Re(q_\nu(x, \xi)) = Re\left(\sum_{\beta} q_\beta^{ij}(x) \xi_\beta\right)$ ($i, j = 1, \dots, r$; ν – die Ordnung des Systems) soll positiv definiert sein für alle $\xi \neq 0$ in jedem $x \in \Omega_n$. Die Anwendbarkeit der Gårding'schen Schlussweise auch bei diesen abgeschwächten Voraussetzungen über die Koeffizienten von Q , sowie die rechte Seite der Gleichung – h , folgt aus dem von mir bewiesenen Hauptsatz in [2].

Wie aus der Bemerkung aus [1] Seite 68 folgt, gilt der folgende

SATZ. *Das verallgemeinerte Neumann'sche Problem (sowie andere dem Riemann'schen Problem entsprechende Randwertaufgaben) können mit Hilfe der Galerkin'schen Methode gelöst werden. Man soll nur die Klasse der elliptischen Gleichungen auf diejenigen einschränken, für die $\sum_{\alpha = \beta = m} p_{\alpha\beta}(x) \xi_\alpha \xi_\beta$ gleichmässig positiv in Ω_n ist.*

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Equations of Motion for Linear Field-Theories

by

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There is, at present, quite a large literature [1], on the derivation of the equations of motion from field equations. Indeed, such a derivation is possible only for non-linear field equations, that is, for gravitational equations. It is more difficult and artificial to formulate the problem for linear theories. I say "to formulate", because the equations of motion cannot be deduced from a linear field theory. For example, consider Maxwell's equations with Lorentz's equations of motion added to them. The equations of motion can be *reformulated*, as Dirac did [2], in terms of field equations only. Yet, we can *deduce* Lorentz's equations of motion by adding those of gravitational field to Maxwell's equations. This is true, because in this way we obtain a *non-linear* system of equations from which the equations of motion can be deduced.

This is rather an artificial procedure since we know that the gravitational field plays a small rôle in the description of microphysical phenomena such as the equations of the electromagnetic or mesonic fields deal with. Yet, the addition of the gravitational equations to the linear field equations will lead us to a very simple formulation of the equations of motion for linear field equations.

Let us, therefore, start with the gravitational equations, which, in the usual notation, are: *)

$$(1) \quad \mathbf{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathbf{R} = -8\pi k \mathbf{T}_{\mu\nu}.$$

Now $\mathbf{T}_{\mu\nu}$ on the right hand side is the energy-momentum tensor density. Let us assume that matter is represented by singularities of the gravitational field and that electromagnetic and mesonic fields are absent. To make things as simple as possible let us also assume that

*) Greek indices run from 0 to 3, Latin indices from 1 to 3. The Galilean values for $g^{\mu\nu}$ are $\eta^{\mu\nu}$; $\eta^{mn} = -\delta^{mn}$; $\eta^{0m} = 0$; $\eta^{00} = 1$.

only *one* singularity is present. In this case we obtain, as was shown in [3], the proper equations of motion by setting

$$(2) \quad \mathbf{T}^{\mu\nu} = m \dot{\xi}^\mu \dot{\xi}^\nu \delta_{(3)}.$$

Here: m is the mass, dependent on time space co-ordinates; the ξ^s are the space co-ordinates of the singularity as functions of time; “dot” means differentiation with respect to time $t = \xi^0$ and $\dot{\xi}^0 = 1$; $\delta_{(3)}$ is the three dimensional Dirac function

$$(3) \quad \delta_{(3)} = \delta(x^1 - \xi^1) \delta(x^2 - \xi^2) \delta(x^3 - \xi^3)$$

Because of Bianchi's identities we have:

$$(4) \quad \mathbf{T}^{\mu\nu} ; ; = 0,$$

where the semicolon denotes covariant differentiation. Thus the equations of motion deduced from (1) with the help of the Bianchi identities [3] are:

$$(5) \quad \frac{dt}{ds} \int \mathbf{T}^{\mu\nu} ; ; d_{(3)}x = 0.$$

Before we calculate (5) and find the equations of the geodetic line from it, let us determine m in (2). We do this using the tensorial condition:

$$(6) \quad \frac{dt}{ds} \int \mathbf{T}^{\mu\nu} d_{(3)}x = m_0 \frac{d\xi^\mu}{ds} \frac{d\xi^\nu}{ds},$$

where m_0 is an invariant, but not necessarily a constant. From (6) and (2) follows

$$(7) \quad m \dot{\xi}^\mu \dot{\xi}^\nu \frac{dt}{ds} = m_0 \dot{\xi}^\mu \dot{\xi}^\nu \left(\frac{dt}{ds} \right)^2$$

or

$$(8) \quad m = m_0 \frac{dt}{ds}.$$

In the case of special relativity theory:

$$(9) \quad m = m_0 (1 - \dot{\xi}^\mu \dot{\xi}^\nu)^{-1/2}.$$

Now, since we know $\mathbf{T}^{\mu\nu}$ in (5), we can go back to these equations and calculate the equations of motion as expressed by (5). The simplest way to do this is to use a local geodetic co-ordinate system, that is one for which — at the point of singularity — the external gravitational field does not exist. In such a co-ordinate system, in the limiting case $m_0 \rightarrow 0$ [4], we find from (5) and (2) the equations

$$(10) \quad \frac{d}{ds} \left(m_0 \frac{d\xi^\mu}{ds} \right) = 0.$$

Multiplying the above equation by $\frac{d\xi^\mu}{ds}$ and using the fact that

$$(11) \quad \frac{d\xi^\mu}{ds} \frac{d^2\xi_\mu}{ds^2} = 0; \quad \frac{d\xi^\mu}{ds} \frac{d\xi_\mu}{ds} = 1,$$

we have the equation

$$(12) \quad \frac{dm_0}{ds} = 0,$$

which states that m_0 must be constant. Thus our equation of motion gives us $\xi^k(t)$ and the information that the mass is a constant. We obtain the well known result that a free particle moves in a gravitational field along a geodetic line. Yet this simple procedure will allow us to formulate the equations of motion for a linear field, say, the electromagnetic or mesonic field.

In the more general case, when besides the gravitational field we also have *another* field, then on the right hand side of (1) is the sum of the mechanical tensor density and that of *another* field:

$$(13) \quad \mathbf{T}^{\mu\nu} = \mathbf{M}^{\mu\nu} + \mathbf{E}^{\mu\nu},$$

where $\mathbf{M}^{\mu\nu}$ is the mechanical tensor density, previously denoted by $\mathbf{T}^{\mu\nu}$, and $\mathbf{E}^{\mu\nu}$ is the tensor density of another (electromagnetic or mesonic) field which, of course, also depends on the gravitational field.

Thus our system of equations is:

$$(14a) \quad \mathbf{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathbf{R} = -8\pi k (\mathbf{M}_{\alpha\beta} + \mathbf{E}_{\alpha\beta})$$

$$(14b) \quad \text{Field equations for the electromagnetic or mesonic field.}$$

Because of Bianchi's identities we have

$$(15) \quad (\mathbf{M}^{\alpha\beta} + \mathbf{E}^{\alpha\beta})_{;\beta} = \mathbf{T}^{\alpha\beta}_{;\beta} = 0,$$

which gives the equations of motion:

$$(16) \quad \frac{dt}{ds} \int (\mathbf{M}^{\mu\nu} + \mathbf{E}^{\mu\nu})_{;\nu} d_{(3)}x = 0.$$

In the case of Special Relativity Theory, that means if we go with $k \rightarrow 0$ and $g_{\alpha\beta} \rightarrow \eta_{\alpha\beta}$, we must reject (14a) and assume (14), (15) and (16) as valid. Thus, in the case of Special Relativity Theory the equations of motion do not follow from the field equations but must be assumed. But it is General Relativity Theory that suggests their form.

Thus for a Cartesian co-ordinate system we have the equations of motion:

$$(17) \quad \frac{dt}{ds} \int (\mathbf{M}^{\mu\nu} + \mathbf{E}^{\mu\nu})_{;\nu} d_{(3)}x = 0.$$

Let us see what these equations tell us about the case of one particle.

Since $M_{\mu\nu}$ has the same form as that given above for $T_{\mu\nu}$, we have

$$(18) \quad \frac{d}{ds} \left(m_0 \frac{d\xi^\mu}{ds} \right) = - \frac{dt}{ds} \int (E^{\mu\nu}{}_{;\nu}) d_{(3)}x .$$

Multiplying by $\frac{d\xi_\mu}{ds}$, we find, as before:

$$(19) \quad \frac{dm_0}{ds} = - \frac{dt}{ds} \frac{d\xi_\mu}{ds} \int E^{\mu\nu}{}_{;\nu} d_{(3)}x ,$$

which shows, that, generally speaking, m_0 will be a function of the proper time. We shall consider later examples of the equations of motion (18) for electromagnetic and mesonic field.

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On Linear Electrodynamics with Higher Derivatives

by

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To remove the well-known difficulties appearing in classical Maxwellian electrodynamics [1] a generalized theory has been developed by Podolski [2]. The generalization consists in the introduction of linear field equations of higher order by means of variational formalism with higher derivatives. For the four-potential one gets in this way a fourth order equation *)

$$(1 - a^2 \square) \square A_a = 0,$$

where a is a constant having the dimension of length, and \square is the d'Alambert operator.

It is well known from [2] that the solution of the equation quoted above is given by the superposition of an ordinary Maxwell field and of a "meson-type" field. This latter field of short range was obtained through the introduction of the second order derivatives to the Lagrange function. The question arises as to which field equations can be obtained through the further generalization of the variational formalism by introducing derivatives of an order higher than second, with the condition that the theory remains linear. The purpose of this note is to answer that question.

*) The following notations will be used in this paper. The Greek indices of tensors run from 1 to 4 ($g_{\alpha\beta} = \delta_{\alpha\beta}$, $x_4 = ict$). A comma denotes differentiation. The pairing of Greek indices denotes summation from 1 to 4. The Latin indices always denote the paired "dummy" indices of summation (also ranging from 1 to 4). They are used in cases where we do not want to indicate the method of pairing the indices. E. g. the expression

$$I = A_{a_1 a_2} B_{b_1 b_2}$$

can denote one of the following three invariants

$$\frac{I}{1} = A_{\alpha\alpha} B_{\beta\beta}, \quad \frac{I}{2} = A_{\alpha\beta} B_{\alpha\beta}, \quad \frac{I}{3} = A_{\alpha\beta} B_{\beta\alpha}.$$

Index r enumerates (in an arbitrarily chosen way) the various possibilities of paring.

Field equations may be obtained from the variational principle

$$(1) \quad \delta \int L d_4x = 0,$$

where the Lagrangian density L depends on the invariants formed from the four-potential A_a , and its derivatives $A_{a,\beta\dots}$ up to and including the n -th order. The Euler equations corresponding to the variational principle (1) are

$$(2) \quad \frac{\partial L}{\partial A_{\gamma_0}} - \left(\frac{\partial L}{\partial A_{\gamma_0\gamma_1}} \right)_{\gamma_1} + \left(\frac{\partial L}{\partial A_{\gamma_0\gamma_1\gamma_2}} \right)_{\gamma_1\gamma_2} - \dots + (-1)^n \left(\frac{\partial L}{\partial A_{\gamma_0\gamma_1\dots\gamma_n}} \right)_{\gamma_1\dots\gamma_n} = 0.$$

The antisymmetrical field tensor $f_{\alpha\beta}$ is defined as usual by the equation

$$(3) \quad f_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}.$$

Since only the field tensor $f_{\alpha\beta}$ has a physical meaning, and not the four-potential A_a , so our theory, and in particular equations (2), must be invariant with respect to the gauge transformation

$$(4) \quad A_a \rightarrow A_a + X_a.$$

Equations (2), i. e. the field equations, according to our assumption, must be linear. Hence it follows that in the case of a field in vacuum (i. e. assuming that there are no charged particles as field sources) the Lagrangian must be of a quadratic form with respect to the field potentials and their derivatives up to and including the n -th order. The coefficients of the form must be constant.

Let us denote by

$$I^{(p,q)} \quad (p, q = 0, 1, \dots, n)$$

an invariant formed from a product of two derivatives of the potential A_a ; one of the derivatives is of the order p , the second of the order q . Index r enumerates all the possible ways of constructing the invariant $I^{(p,q)}$

from a product of two derivatives of A_a , of the order p and q respectively. The range of variation of the index r we shall fix later. So, the part of the Lagrangian arising from invariants formed from the derivatives of A_a , of the order p and q , is of the form

$$\sum_r b^{(r)} I^{(p,q)},$$

where $b^{(r)}$ are coefficients independent of the four-point (we shall see below that it is also not necessary to assume them to be dependent on the indices p and q).

So, the total Lagrangian will be given by the sum

$$(5) \quad L = \sum_{p,q=0}^n a^{(p,q)} \sum_r b^{(r)} I^{(p,q)},$$

where $a^{(00)} = 0$ according to our assumption of the invariance with respect to the gauge transformation. The coefficients $a^{(p,q)}$ are of course independent of x_μ .

Now we shall consider the invariant $I^{(p,q)}$. We have

$$(6) \quad I^{(p,q)} = \int A_{a_0 a_1 \dots a_p} A_{b_0 b_1 \dots b_q}.$$

Since $I^{(p,q)}$ is an invariant, it follows that the number $p+q+2$ must be an even one. So we may introduce the notation

$$s = (p+q)/2,$$

where s is an integer.

Now we shall investigate which terms arising from the invariant $I^{(p,q)}$ will appear in the field equations (2). Two such terms may be found:

$$\begin{aligned} P_{\gamma_0} = (-1)^p \left(\frac{\partial I^{(p,q)}}{\partial A_{\gamma_0 \gamma_1 \dots \gamma_p}} \right)_{\gamma_1 \dots \gamma_p} &= (-1)^p \{ A_{b_0 b_1 \dots b_q} \cdot \delta_{a_0 \gamma_0} \cdot \delta_{a_1 \gamma_1} \dots \delta_{a_p \gamma_q} \}_{\gamma_1 \dots \gamma_q} = \\ &= (-1)^p A_{b_0 b_1 \dots b_q a_1 \dots a_p} \cdot \delta_{a_0 \gamma_0}, \end{aligned}$$

(where δ_{ab} is the usual Kronecker symbol), and

$$\bar{P}_{\eta_0} = (-1)^q \left(\frac{\partial I^{(p,q)}}{\partial A_{\eta_0 \eta_1 \dots \eta_q}} \right)_{\eta_1 \dots \eta_q} = (-1)^q A_{a_0 a_1 \dots a_p b_1 \dots b_q} \cdot \delta_{b_0 \eta_0}.$$

The expression P_{γ_0} and \bar{P}_{η_0} are four-vectors. First, we shall consider the vector P_{γ_0} . Two cases are possible which will be distinguished by the indices $r=1$ and $r=2$ respectively. Namely

1) The indices a_0 and b_0 are paired, i. e.,

$$P_{\gamma_0} = (-1)^p A_{a_1 a_2 \dots a_p b_1 \dots b_q} \cdot \delta_{a_0 \gamma_0},$$

then

$$P_{\gamma_0} = (-1)^p A_{a_1 a_2 \dots a_p b_1 \dots b_q},$$

and as the indices after the comma are indices of summation, so

$$(7) \quad P_{\gamma_0} = (-1)^p \square^{\frac{p+q}{2}} A_{\gamma_0} = (-1)^p \square^s A_{\gamma_0}.$$

2) The indices a_0 and b_0 are not paired. Then in the expression

$$P_{\gamma_0} = (-1)^q A_{a_1 a_2 \dots a_p b_1 \dots b_q} \delta_{a_0 \gamma_0}$$

one of the indices after the comma is paired with b_0 , and one with a_0 . The remaining indices are paired among themselves. So

$$(8) \quad \underset{2}{P}_{\eta_0} = (-1)^p \square^{s-1} A_{\varrho, \varrho \eta_0}.$$

Since there are no other combinations (because the pairing and non-pairing of the indices a_0 and b_0 comprise all possibilities), so index r can have only two values: 1 and 2.

In a similar way we find for $\underset{r}{P}_{\eta_0}$ two possibilities:

$$\underset{1}{\bar{P}}_{\eta_0} = (-1)^q \square^s A_{\eta_0}$$

$$\underset{2}{\bar{P}}_{\eta_0} = (-1)^q \square^{s-1} A_{\varrho, \varrho \eta_0}.$$

But since $p+q=2s$, so $(-1)^p=(-1)^q$, and hence

$$(9) \quad \underset{1}{P}_a = \underset{1}{\bar{P}}_a \quad \text{and} \quad \underset{2}{P}_a = \underset{2}{\bar{P}}_a.$$

All terms arising from the invariant $\underset{r}{I}^{(p,q)}$ ($r=1,2$, and p,q fixed) give the expressions

$$(10) \quad \underset{1}{2} (P_{\eta_0} b^{(1)} + \underset{2}{P}_{\eta_0} b^{(2)}) a^{(p,q)}.$$

So, without losing the generality of our considerations, we may write the Lagrangian (5) in the form

$$(11) \quad L = \sum_{p,q=0}^n a^{(p,q)} \{ b^{(1)} \underset{1}{I}^{(p,q)} + b^{(2)} \underset{2}{I}^{(p,q)} \}.$$

Now, from the invariance of the field equations with respect to the gauge transformation we shall determine the coefficients $b^{(1)}$ and $b^{(2)}$.

Substituting

$$A_a = \varphi_a + \chi_{,a},$$

expression (10) must remain invariant and hence the terms containing the arbitrary function must vanish. Using the formulas (7), (8), (9), we get

$$\underset{1}{P}_a b^{(1)} + \underset{2}{P}_a b^{(2)} = (-1)^p \{ b^{(1)} \square^s A_a + b^{(2)} \square^{s-1} A_{\beta, \beta a} \} =$$

$$= (-1)^p \{ b^{(1)} \square^s \varphi_a + b^{(2)} \square^{s-1} \varphi_{\beta, \beta a} \} + (-1)^p (b^{(1)} + b^{(2)}) \square^s \chi_{,a}.$$

Hence

$$(-1)^p (b^{(1)} + b^{(2)}) \square^s \chi_{,a} = 0,$$

and consequently

$$b^{(1)} = -b^{(2)}.$$

But since only the ratio of the coefficients $b^{(1)}$ and $b^{(2)}$ can be determined (by reason of the arbitrariness of the coefficients $a^{(p,q)}$), we may assume

$$(12) \quad b^{(1)} = 1, \quad b^{(2)} = -1.$$

Consequently, the final form of the Lagrangian will be given by

$$(13) \quad L = \sum_{p,q=0}^n a^{(p,q)} \{ I_1^{(p,q)} - I_2^{(p,q)} \},$$

and the field equations following from (13) will take the general form

$$(14) \quad (a_0 + a_1 \square + a_2 \square^2 + \dots + a_m \square^m) f_{\alpha\beta,\beta} = 0, \quad (m = n - 1),$$

because, owing to equalities (12), we have

$$\begin{aligned} \underset{1}{P}_a b^{(1)} + \underset{2}{P}_a b^{(2)} &= (-1)^p \{ \square^s A_a - \square^{s-1} A_{\beta,\beta a} \} = \\ &= (-1)^p \square^{s-1} (A_{a,\beta} - A_{\beta,a})_{,\beta} = (-1)^{p+1} \square^{s-1} f_{a\beta,\beta}. \end{aligned}$$

Since a polynomial of the m -th order has m zeros, the equation (14) may be written in the form

$$(15) \quad \prod_{j=1}^m (\square - \mu_j^2) f_{a\beta,\beta} = 0,$$

where μ_j^2 denotes the roots (generally complex) of the polynomial from equation (14).

Instead of the usual Lorentz auxiliary condition let us assume a much less restrictive condition of the form

$$(16) \quad \prod_{j=1}^m (\square - \mu_j^2) A_{a,a} = \text{const.}$$

Then the wave equation corresponding to (15) will be given by

$$(17) \quad \prod_{j=1}^m (\square - \mu_j^2) A_a = 0.$$

In the case of a non-moving "point-electron" (i. e. in the case of a static field of spherical symmetry) we have, instead of equation (17), the equation

$$(18) \quad \prod_{j=1}^m (\Delta - \mu_j^2) \Delta \Phi = 0$$

with a singularity in the origin of the co-ordinate system. Δ denotes, as usual, the Laplacian operator, and

$$\Phi(r) = \frac{1}{i} A_4$$

— the scalar potential.

The solutions of equation (18) will be given by

$$(19) \quad \Phi = \frac{1}{r}, \quad \Phi = \frac{e^{-\mu_j r}}{r} \quad (j = 1, 2, \dots, m).$$

When $\mu_i \neq \mu_j$ for $i \neq j$ ($i, j = 1, 2, \dots, m$) and $\mu_j \neq 0$ the solutions (19) are linearly independent, and every solution of equation (18) is given by a superposition of the solutions (19). If equation (14) has zero-, and multiple-roots we get, besides, additional solutions of the form $\Delta^2 \Phi = 0$, $(\Delta - \mu_j^2)^s \Phi = 0$.

The quantum theory of equations of the type (17) was first given by Pais and Uhlenbeck [3]. They use this equation as a starting point for their calculations.

The purpose of the present note is to show that linear variational formalism with higher derivatives (of finite order) is not able to give anything more than an equation of the type (17).

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The Electric Conductivity and Catalytic Activity of $ZnO \cdot Fe_2O_3$ Mixtures

by

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Presented by B. KAMIEŃSKI on January 20, 1954

It is well known that the electric conductivity of semiconducting oxides and sulphides is influenced by the chemisorption processes taking place on their surface.

The literature relating to the influence of oxygen pressure on the conductivity of oxides is given in the papers published by Anderson [1] and Bevan and Anderson [2]. Garner, Gray and Stone [3] have studied the changes in the conductivity of copper oxide interacting at its surface with oxygen, hydrogen or carbon monoxide. All these investigations show that the chemisorption of gases by semiconductors is accompanied by electronic processes resulting in the transition of a certain number of electrons to the conductivity band or in the formation of some positive holes in the normal band or at local levels. According to Engell [4] such electronic changes play an essential rôle in catalytic processes. Volkenstein [5] has discussed, from the theoretical point of view, the influence of the electronic processes taking place in a semiconductor on its catalytic activity.

Although the rôle of electronic processes in catalysis is very often emphasized and the electric conductivity of many semiconducting catalysts has been studied, no measurements of the electric conductivity of such catalysts in the course of their catalytic reaction have yet been made. In the present investigation we undertook such measurements, using a mixture of ZnO and Fe_2O_3 as catalyst for the catalytic oxidation of ethyl alcohol. The same catalyst was applied by Miyata [6] in the course of his study of the catalytic conversion of ethyl alcohol to acetone. The electric conductivity of $ZnO \cdot Fe_2O_3$ mixtures (in air) has been investigated by two of the present authors [7]. The reagents used were chemically pure ferric oxide (Gliwice) and pharmacopoeial zinc oxide. Spectral analysis showed no impurities in the ferric oxide and traces

(about 0.01%) of MgO in the zinc oxide. Both oxides were mixed in the stoichiometric ratio of 1:1, and 2 g. pellets (diameter = 12 mm., height = 6 mm.) were pressed out with a hand press. Five pellets were used for one experiment. They were put in a quartz tube (diameter = 35 mm.) and then placed in an electric furnace with temperature control. One of the pellets was pressed between two ceramic plates, on which there were two electrodes of thin platinum sheet. The quartz tube was connected at its upper end to a glass bulb kept in an oil bath at 120°. A mixture of ethyl alcohol was dropped at a constant rate into the bulb by means of a capillary. The evaporation of each small portion of the liquid was completed before a new portion was introduced in order to prevent any change in the composition of vapours due to fractional distillation. The lower end of the quartz tube was connected to a vacuum pump through a water-cooled condenser and a freezing condenser. The temperature was measured by means of a Pt-PtRh thermocouple, the end of which was situated immediately adjacent to the sample.

In our conditions we found, besides alcohol and water, only acetaldehyde in the condensation products. It is probable that the small quantity of the catalyst and the relatively short time of its contact with the reacting vapour caused no further stages to take place in the reaction and so no acetone was formed. The progress of the reaction was checked by volumetric determination of the acetaldehyde with the help of hydroxylamine hydrochloride [8].

The electric conductivity was measured with the help of an a. c. linear bridge with amplification. The method has been described by Jander, Tubandt and Lorenz. According to Lorenz [9] the inclusion of an amplifier in the system not only enables the range of the measurements to be enlarged but also contributes to an improvement in sensitivity. Before starting the reaction, the air was pumped off from the apparatus down to a pressure of about 1 mm. Hg. There was always quite an appreciable change in the conductivity of the catalyst after the vapours of the water-alcohol mixture had been introduced into the apparatus. After about 5 minutes the conductivity reached a new constant value. According to the conditions of the experiment, the change in the conductivity logarithm amounted to 1-5 units. When the current of flowing vapours was cut off and air again introduced at a pressure of 1 mm. Hg. the conductivity dropped back to its initial value. The course of these processes is shown in Fig. 1. The curves correspond to temperatures of 300°, 450° and 550°. A 20% (vol.) alcohol-water mixture was used. The moment, when the vapours were let into the apparatus, was taken as zero time in the diagram. The introduction of pure water vapour did not affect the conductivity of the catalyst.

In the course of our investigation we could not find any distinct relation between the initial conductivity of the catalyst (the conductivity

measured before the water-alcohol mixtures were introduced) and its catalytic activity. Neither did Voltz and Weller [10] find any direct connection in the case of hydrogen deuterium exchange on a chromic

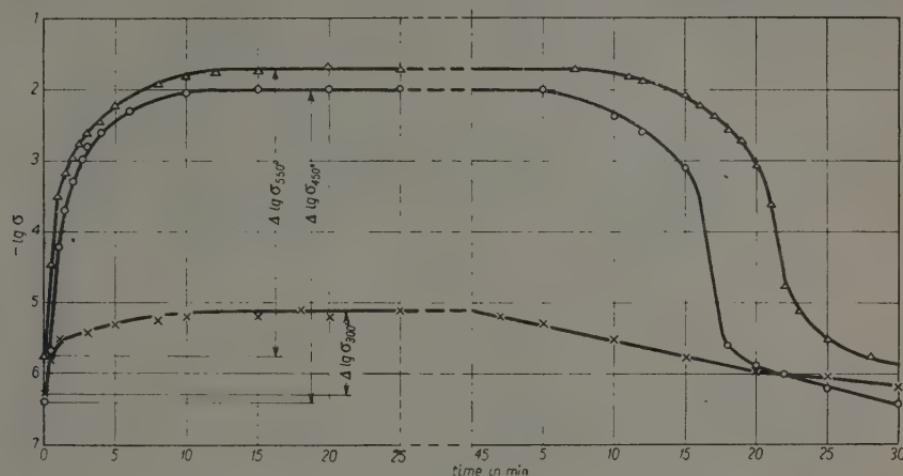


Fig. 1

oxide catalyst. We found, however, quite a distinct parallel between the catalytic activity of our catalyst and the change in its conductivity taking place when the reaction substrates are introduced. We expressed these changes as the difference between logarithms of the initial and final conductivity of the catalyst ($\Delta \log \sigma$). The $\Delta \log \sigma$ values for the nonsintered catalyst and 20% (vol.) water-alcohol mixture are plotted in Fig. 2. (curve I) as a function of the reaction temperature. Curve II shows the reaction yield at the same temperatures. It is expressed in mg. of CH_3CHO per 25 ml. of the water-alcohol mixture. Owing to the small reaction yield in

our experiment, the temperature of the sample could only be slightly influenced by the heat evolved in the course of the reaction. At 700° the resistance of the sample was so low, that it could not be determined with satisfactory precision with our apparatus. The results show quite a distinct parallel between the $\Delta \log \sigma$ value and the catalytic activity

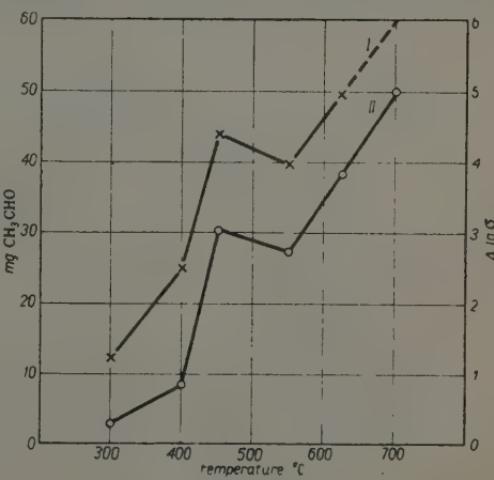


Fig. 2

of $\text{ZnO-Fe}_2\text{O}_3$ mixtures. The curves show a local maximum of about 450° , which agrees well with the results of Miyata, who found this temperature optimal.

Table I shows the influence of alcohol concentration on the $\Delta \log \sigma$ value and catalytic activity of the same catalyst at 450° .

TABLE I

Influence of alcohol concentration on $\Delta \log \sigma$ value and catalytic activity of $\text{ZnO-Fe}_2\text{O}_3$ mixtures

Concentration of alcohol (% vol.)	$\Delta \log \sigma$	Reaction yield (mg. CH_3CHO)
10	3.2	4.4
20	4.6	30.7
48	3.8	21.6

We found the highest reaction yield when using a 20% alcohol-water mixture (again in accordance with Miyata's results). This also corresponded to the highest $\Delta \log \sigma$ value.

The best catalytic activity and also the highest $\Delta \log \sigma$ value were found in the case of non-sintered catalysts. Catalysts sintered at 700° gave lower results and samples sintered at 900° or 1200°C . showed only slight catalytic activity and a very low $\Delta \log \sigma$. The numerical results for 20% water-alcohol mixtures at 450° are given in Table II.

TABLE II

Influence of sintering on the catalytic activity of $\text{ZnO-Fe}_2\text{O}_3$ mixtures and $\Delta \log \sigma$ value at 450°C (20% alcohol-water mixture)

Catalyst	Reaction yield (mg. CH_3CHO)	$\Delta \log \sigma$
non-sintered catalyst	30.7	4.6
sintered catalyst at 700°	9.7	2.2
sintered catalyst at 900°	5.3	0.3
sintered catalyst at 1200°	4.0	0.05

Our results show that there exists a distinct parallel between the catalytic processes taking place on the surface of the catalyst and the changes in its electrical conductivity observed in the course of the catalytic reaction.

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The Absorption Spectrum of KCl Crystals Containing Metallic Thallium Centres

by

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Presented by M. ŚMIAŁOWSKI on January 22, 1955

Introduction

In a previous paper [1] a description was given of the photoelectric properties of KCl crystals containing metallic thallium centres. Information derived from the spectral distribution of the photo-effect and from the observed behaviour of the photo-electric current have led to the inference that thallium centres prepared in the manner described are atomically dispersed throughout the crystal.

In order to test this inference and to obtain further information on the nature of the centres under investigation, the absorption constant of the crystals was investigated.

The absorption spectrum of KCl crystals, containing *F* centres and also small amounts of Tl ions, has already been measured by O. Stasiw [2], but his samples were prepared in different ways so that there were differences both in the contents and in the nature of the centres, and therefore his results cannot be used for determining the nature of the centres investigated in the present work.

Experimental work

The absorption constant was measured photo-electrically [3] by means of a single monochromator and a Wulf electrometer (sensitivity about 20 mV/dash) registering the photo-current of a vacuum photo-cell. A tungsten lamp and a mercury lamp were used as sources of light for the visible and ultraviolet regions respectively. In order to estimate the losses caused by the reflection of light on the crystal surface, the absorption of the crystal under examination was compared with that of pure KCl crystal.

Results and discussion

The absorption spectrum in the visible region of a KCl crystal, containing $(4.1 \pm 0.14) \cdot 10^{-3}$ per cent mol Tl, is shown in Fig. 1, curve *a*. It can be seen that the absorption constant rises with an increase in the frequency but does not pass the maximum.

The absorption spectrum in the ultraviolet region is shown in Fig. 2. Curve *a* refers to the same sample as indicated by curve *a* in Fig. 1; curve *b* refers to another sample of crystal with the same thallium content. In both cases a maximum was observed at $\lambda_{\max} = (309 \pm 1) \text{ m}\mu$.

An "Algérie" spectrophotometer, made by A. Jobin et G. Yvon, was used for measuring the absorption constant over a wider ultraviolet range; the spectral distributions obtained are shown in Fig. 3 *). Curve *a*

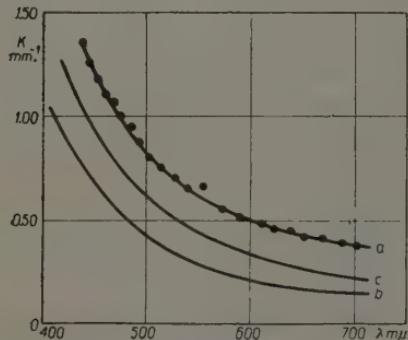


Fig. 1

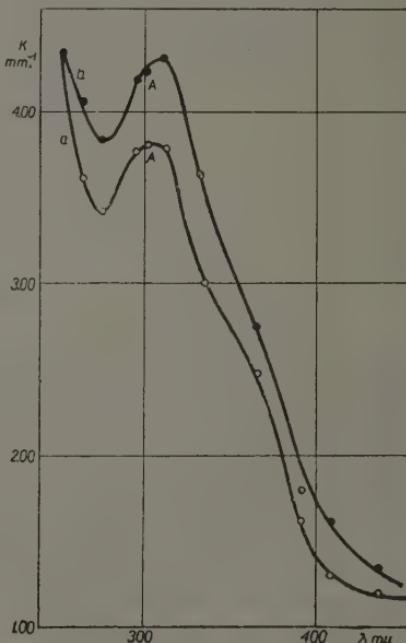


Fig. 2

refers to a crystal containing $(4.1 \pm 0.14) \cdot 10^{-3}$ per cent mol Tl and curve *b* to a crystal with a content of $(6.2 \pm 0.35) \cdot 10^{-3}$ per cent mol Tl. Two absorption peaks were observed; band *A* at $(309 \pm 1) \text{ m}\mu$ and $300 \text{ m}\mu$, and band *B* at $238 \text{ m}\mu$ and $270 \text{ m}\mu$. Curve *c* refers to a crystal containing $2.1 \cdot 10^{-2}$ per cent mol Tl plus a certain excess of potassium atoms. In this case a very broad absorption band is observed.

From the absorption peak of curve *a* or *b* in Fig. 2 we may obtain the concentration of absorbing centres according to the formula:

$$(1) \quad N = 1.31 \cdot 10^{17} \cdot \frac{n_0}{(n_0^2 + 2)^2} K_{\max} H \frac{\omega + \frac{\mu}{2}}{\omega + \mu},$$

*) The author would like to thank H. Kozdaj and W. Mański for enabling him to carry out this part of the measurements in the Hirschfeld Institute of the Polish Academy of Sciences in Wrocław.

where N denotes the concentration of absorbing centres in cm.^{-3} , n_0 — the refraction index of alkali halide crystal for the frequency ω , K_{\max} — the value of the absorption constant for the maximum of absorption in cm.^{-1} , H and μ — the "half-width" of the absorption peak measured in eV. and sec.^{-1} , respectively, and ω — the frequency corresponding to the maximum in sec.^{-1} . Expression (1) was derived by A. Smakula [4] from the classical dispersion theory, and was used by R. W. Pohl and his collaborators [4], [5] for similar calculations of F centres concentration. It can be shown by detailed discussion that expression (1) may be used equally well for determining the concentration of thallium centres.

The values needed for calculation of the number N may be read from curve a in Fig. 2. These are:

$$\omega_1 = 6.16 \cdot 10^{15} \text{ sec.}^{-1},$$

$$K_{\max} = 38.2 \text{ cm.}^{-1}$$

$$\mu_1 = 1.24 \cdot 10^{15} \text{ sec.}^{-1},$$

$$H = 1.64 \text{ eV.}$$

The value of $n_0 = 1.54$ is found by extrapolation to the frequency ω . Substituting the numbers given above in expression (1) we thus obtain:

$$N_1 = 6.1 \cdot 10^{17} \text{ cm.}^{-3}.$$

From the analytical content of thallium $(4.1 \pm 0.14) \cdot 10^{-3}$ per cent mol, we can calculate the concentration of thallium ions present in the crystal before the process of reduction into thallium atoms takes place.

The concentration is:

$$N'_1 = 6.4 \cdot 10^{17} \text{ cm.}^{-3}.$$

A similar calculation can also be made by using the values from the absorption peak of curve b in Fig. 3. In this case we obtain from expression (1):

$$N_{11} = 1.0 \cdot 10^{18} \text{ cm.}^{-3}$$

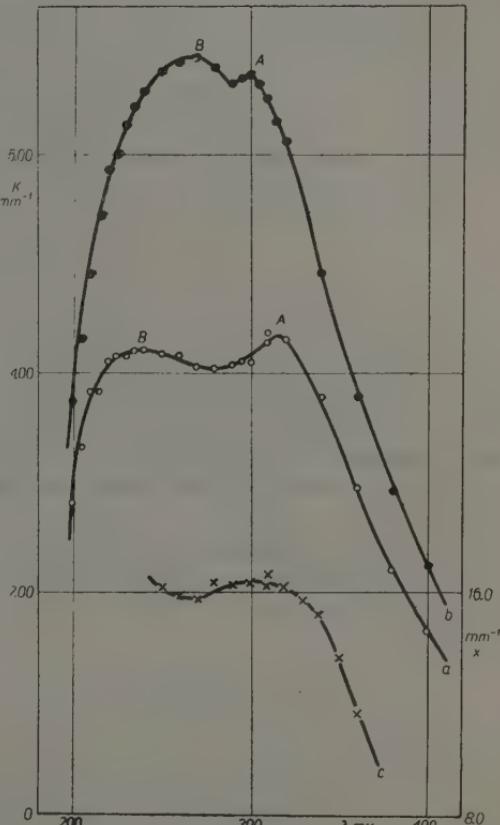


Fig. 3

and from the analytical content:

$$N'_{11} = 1.0 \cdot 10^{18} \text{ cm.}^{-1}.$$

The agreement obtained in this manner between N_1 and N_{11} , and N'_1 and N'_{11} is very good and seems to constitute strong evidence for the atomic dispersion of thallium centres, as well as being a further proof that the absorption band A is quantitatively caused by the presence of atomically dispersed thallium atoms.

In addition, an expression for the dependence of the absorption constant on the frequency of the absorbed light in the visible part of the spectrum can also be deduced from the classical dispersion theory.

Detailed calculation, based upon the assumption that there exists only one absorption peak (band A) in the ultraviolet, leads to the formula:

$$(2) \quad K = \frac{185 \cdot 10^{30} \omega^2}{(\omega_1^2 - \omega^2)^2 + 4.85 \cdot 10^{30} \omega^2},$$

which is represented by curve b in Fig. 1. It is evident that the calculated values are too low in comparison with those observed, but the shape of curve a is similar to that of b .

A better agreement between both curves can be obtained if the other absorption band, B , is also taken into account. In order to simplify the calculation, it was assumed that band B is caused by the presence of the same kind of absorption centres in the same concentrations as in the case of band A . The result is represented by curve c in Fig. 1.

The author wishes to thank Professor K. Gumiński for his constant interest in the work.

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Phase Diagrams for the Production of Silicate Thermophosphates

by

W. BOBROWNICKI and Z. ŚWIECKI

Presented by W. BOBROWNICKI on January 12, 1954

I. The purpose of this work was to examine the oxide phase system for those oxides which are a component part of magnesium thermophosphate fertilisers. Since fertilisers of this type contain as primary components MgO , CaO , P_2O_5 , SiO_2 , as well as CaF_2 , not counting minor quantities of impurities such as Fe_2O_3 , FeO , TiO , etc., this allows a more accurate explanation to be given of the structure and formation of the thermophosphates of a five-component diagram. The contemporary literature deals only with the systems:

$MgO-SiO_2$ [1], $CaO-SiO_2$ [2], and $CaO-P_2O_5$ [3],

or with some ternary systems such as

$CaO-MgO-SiO_2$ [4], or $CaO-P_2O_5-SiO_2$ [5].

Systems of calcium phosphates with magnesium silicates, closely connected with the production of magnesium thermophosphates, are unknown.

II. Our investigation consisted in determining the melting points within the range of the phase diagrams $2MgO \cdot SiO_2 - 3CaO \cdot P_2O_5$, $2MgO \cdot SiO_2 - 3Ca_3(PO_4)_2 \cdot CaF_2$, $MgO \cdot SiO_2 - 3Ca_3(PO_4)_2 \cdot CaF_2$. The melting points were determined in a cryptol furnace on cones made of the examined samples in comparison with pyrometric cones of known melting points. Melting points obtained by this method were liable to experimental error of $\pm 20^\circ C$. and the melting point curves given, being in fact curves representing the deformation temperature of the cones, should be taken as lines lying between the liquidus and solidus curves on exact phase diagrams. A further error might be caused by the action of the reducing atmosphere inside the furnace which would eventually remove

a certain amount of phosphate from the samples under examination. This was to a certain degree avoided by reducing to a minimum the time needed for the determination of the melting points. Consideration of the diagrams thus obtained need not give rise to doubts of a technical nature as to the results, since the production of magnesium thermophosphate on a technical scale takes place in a reducing atmosphere.

Chemically pure magnesium carbonate, calcium carbonate, ammonium phosphate-silicate and calcium fluoride were used to make up the samples for the measuring of melting points. Suitable mixtures of those substances were made and fired for six hours at 1000° C. in an oxidising atmosphere. After being broken up, the mixtures were sieved through a 0.09 mm. mesh sieve (as a whole), and once more homogenised by grinding in a porcelain mortar. After this the mixtures were transformed into a doughlike mass by the addition of a 3% starch solution. This mass was then shaped into cones for the determination of melting points.

III. The results obtained are shown in the following figures:

The difference to be observed in the course of the melting point curve of the forsterite-tricalcium phosphate system and in that of the

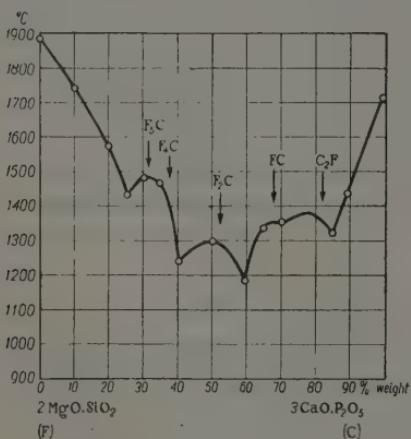


Fig. 1

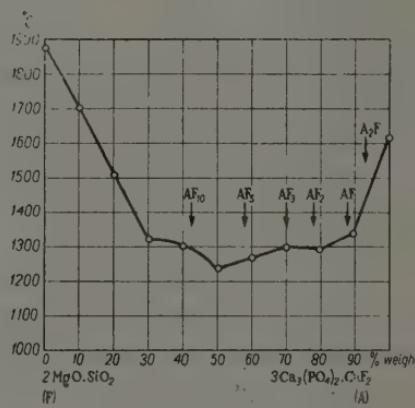
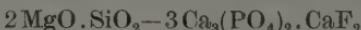


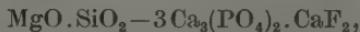
Fig. 2

forsterite-apatite system is all the more striking since the two systems differ only by a small addition of CaF₂.

The sharp break in the melting point curves in the border region of the diagrams:



and



suggests the formation of crystalline solutions with a gap of miscibility, which would point to the apatite structure remaining unchanged by

the introduction of up to 10% per weight of forsterite or 30% per weight of $\text{MgO} \cdot \text{SiO}_2$, (clinoenstatite or enstatite). Whether such a supposition is correct would have to be shown by microscopic and X-Ray examinations of samples of the composition coming within the range of the diagrams. Such tests are at present being carried out.

It should be emphasised that the lowest melting point hitherto obtained by examining these systems was 1180°C ., and it seems unlikely that a lower melting point could be obtained for any mixture included in the pseudoternary system $2\text{MgO} \cdot \text{SiO}_2 - \text{MgO} \cdot \text{SiO}_2 - 3\text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaF}_2$, since, as can be seen from Fig. 4, the pseudoternary phase system is contained in 3 double-systems, of which only two form binary eutectics. On the other hand, there appears to be such a possibility for the pseudoternary diagram $\text{MgO} \cdot \text{SiO}_2 - \text{SiO}_2 - 3\text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaF}_2$. It is,

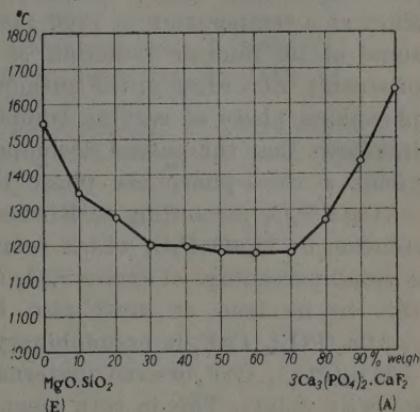


Fig. 3

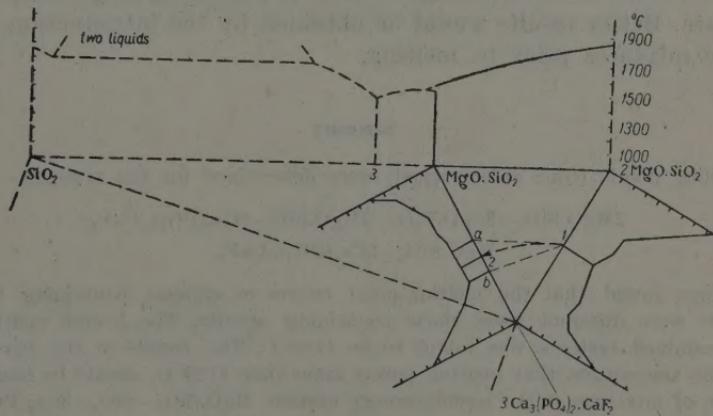


Fig. 4

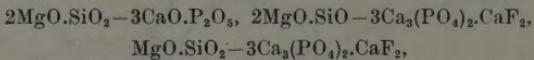
however, extremely likely that the eutectic line passes through the pseudoternary system $2\text{MgO} \cdot \text{SiO}_2 - \text{MgO} \cdot \text{SiO}_2 - 3\text{Ca}(\text{PO}_4)_2 \cdot \text{CaF}_2$, like line 1 → 2 in Fig. 4 (it can be seen that line 1 → 2 should lie along the points 1 → a → b; its precise position can only be determined by thermic tests of the pseudoternary system under consideration, together with microscopic examination of the system $\text{MgO} - 3\text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaF}_2$), which means that in this diagram the existence of ternary eutectics cannot be expected.

On the other hand, in the pseudoternary system $\text{MgO} \cdot \text{SiO}_2 - \text{SiO}_2 - 3\text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaF}_2$ the analogical eutectic line $3 \rightarrow 2$ would only appear if apatite and silica always formed a crystalline solution, which is unlikely. The experiments of Volkovich, Ilarionov and Riemien [6] show that silica at a temperature of $1300-1400^\circ\text{C}$. reacts on the apatite by removing some of its fluoride (reaction is much more intensive in the presence of steam). Accordingly, the authors suggest the formation of a new silica-phosphate phase of varying composition (crystalline solution). From this it follows that the partial decomposition of the apatite with silica finally yields a silica-phosphate phase included in the phase diagram $\text{SiO}_2 - 3\text{CaO} \cdot \text{P}_2\text{O}_5$ (according to Körber and Trömel [3] this is the crystalline solution $\alpha - 3\text{CaO} \cdot \text{P}_2\text{O}_5$ which remains unchanged by the introduction of a small percentage of excess SiO_2 and CaO) as well as apatite. Admitting this to be true, it must also be accepted that the system $\text{SiO}_2 - 3\text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaF}_2$ is pseudobinary, in reality ternary $\text{SiO}_2 - 3\text{CaO} \cdot \text{P}_2\text{O}_5 - 3\text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaF}_2$ or even quaternary $\text{SiO}_2 - \text{P}_2\text{O}_5 - \text{CaO} - \text{CaF}_2$ and contains eutectic points. This in turn means that the mixtures included within the diagram $\text{SiO}_2 - \text{MgO} \cdot \text{SiO}_2 - 3\text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaF}_2$ should have melting points below 1180°C ., corresponding to the actual eutectics (or very similar to them) of much more complicated systems.

Therefore the melting of apatite with serpentine alone does not appear to be the most advantageous way of producing magnesium thermophosphate. Better results would be obtained by the introduction of silica into the mixtures prior to melting.

Summary

Melting points (cone deformation) were determined for the systems



It was found that the melting-point curves of systems containing tricalcium phosphate were different from those containing apatite. The lowest melting point of the examined systems was found to be 1180°C . The results of the investigation permit the assumption that melting points lower than 1180°C . should be found within the range of mixtures of the pseudoternary system $\text{MgO} \cdot \text{SiO}_2 - \text{SiO}_2 - 3\text{Ca}_3(\text{PO}_4)_2 \cdot \text{CaF}_2$.

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